

Integration by parts and convergence in  
distribution norms in the *CLT*

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**CLT**  $X_i \in R^d, i \in N,$  *i.i.d.* with  $E(X_i) = 0, E(X_i^k X_i^p) = \delta_{k,p}$

$$E(\phi(\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i)) \rightarrow \int \phi(y) \gamma(y) dy$$

$\phi \in C_b \rightarrow$  weak convergence

$\phi$  measurable bounded  $\rightarrow$  convergence in total variation

**THEOREM (Prohorov)** Convergence in total variation is equivalent to : there exists  $m$  such that

$$Y_1 = X_1 + \dots + X_m \sim f(x)dx + \nu(dx).$$

**Remark**

$$Z = Y_1 + Y_2 \sim g(x)dx + \mu(dx) \quad g \in C(R)$$

## Consequence

$$g \geq \varepsilon \mathbf{1}_{B_r(x_0)} \geq \varphi_r(x - x_0) \quad \varphi_r \in C^\infty$$

## Splitting

$$Z \sim \xi \times V + (1 - \xi)U$$

With :  $\xi, V, U$  independent with laws

$$P(V \in dx) = \frac{1}{c_r} \varphi_r(x - x_0) dx \quad \varphi_r \simeq \mathbf{1}_{B_r(0)}$$

$$P(\xi = 1) = \varepsilon c_r, \quad P(\xi = 0) = 1 - \varepsilon c_r$$

$$P(U \in dx) = \frac{1}{1 - \varepsilon c_r} (P(Z \in dx) - \varepsilon \varphi_r(x - x_0) dx)$$

**Strategy** : we use an "abstract Malliavin calculus" based on  $V_1, V_2, \dots, V_n, \dots$  in order to prove Prohorov's Theorem.

**Convergence in total variation.**

$$\left| E(f(S_n)) - \int f(x)\gamma(x)dx \right| \leq \frac{C}{\sqrt{n}} \|f\|_\infty$$

and moreover **Convergence in (almost) distribution norms** : let  $|\alpha| = q$ .

$$\left| E(\partial_\alpha f(S_n)) - \int \partial_\alpha f(x)\gamma(x)dx \right| \leq \frac{C}{\sqrt{n}} \|f\|_\infty + e^{-cn} \|f\|_{q,\infty}$$

with

$$\|f\|_{q,\infty} = \sum_{|\alpha| \leq q} \|\partial_\alpha f\|_\infty.$$

**Corollary**

$$\gamma_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{1/2}} e^{-\frac{|x|^2}{2\varepsilon}}$$

Then, with  $\varepsilon_n = e^{-c'n}$

$$\left| E(\partial_\alpha f * \gamma_{\varepsilon_n}(S_n)) - \int \partial_\alpha f(x)\gamma(x)dx \right| \leq \frac{C}{\sqrt{n}} \|f\|_\infty$$

**Corollary** If  $Z_i \sim p(x)dx$ ,  $p \in W^{1,1}$  ( $\nabla p \in L^1$ ) Then  $S_n \sim p_{S_n}(x)dx$  and

$$\left\| \partial^\alpha p_{S_n} - \partial^\alpha \gamma \right\|_\infty \leq \frac{C}{\sqrt{n}} \quad \forall |\alpha| \leq n$$

## Extensions

A.  $Z_i, i \in N$  are **not identically distributed**

B **Local theorems (Edgeworth expansions)**

$$\left| E(\partial_\alpha f(S_n)) - \int \partial_\alpha f(x) \left( \sum_{i=0}^k \frac{1}{n^{i/2}} \theta_i(x) \right) \gamma(x) dx \right| \leq \frac{C}{n^{(k+1)/2}} \|f\|_\infty + e^{-cn} \|f\|_{q,\infty}$$

## Expected number of roots for trigonometric polynomials

$$P_n(t, Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^n (Y_k^1 \cos \frac{kt}{n} + Y_k^2 \sin \frac{kt}{n})$$
$$N_n(Y) = \text{card}\{t \in (0, n\pi) : P_n(t, Y) = 0\}.$$

## Invariance principle

$$\lim_n \frac{1}{n} E(N_n(Y)) = \lim_n \frac{1}{n} E(N_n(G)) = \frac{1}{\sqrt{3}}$$

**Kac-Rice lemma** let  $f : [0, a] \rightarrow R$  differentiable such that

$$\inf_{t \leq a} (|f(t)| + |f'(t)|) > 0.$$

Then

$$N_a(f) = \lim_{\delta \rightarrow 0} \int_0^a |f'(t)| \mathbf{1}_{\{|f(t)| \leq \delta\}} \frac{dt}{2\delta}.$$

We use the *CLT* for  $S_n = (S_n^1, S_n^2) \in R^2$  with

$$S_n^1(t, Y) = P_n(t, Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^n (Y_k^1 \cos \frac{kt}{n} + Y_k^2 \sin \frac{kt}{n})$$

$$S_n^2(t, Y) = \partial_t P_n(t, Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{k}{n} (Y_k^2 \cos \frac{kt}{n} - Y_k^1 \sin \frac{kt}{n})$$

We take

$$\Phi_\delta(x^1, x^2) \simeq |x_2| \times \mathbf{1}_{(-\infty, 0)}^{(\delta)}(x^1) \quad \rightarrow \quad \partial_1 \Phi_\delta(x_1, x_2) \approx x_2 \delta_0(x_1)$$

K-R lemma

$$N_n(Y) = \lim_{\delta \rightarrow 0} \int_0^{n\pi} \partial_1 \Phi_\delta(S_n^1(t, Y), S_n^2(t, Y)) dt$$

CLT

$$\lim_n E(\partial_1 \Phi_\delta(S_n^1(t, Y), S_n^2(t, Y))) = \lim_n E(\partial_1 \Phi_\delta(S_n^1(t, G), S_n^2(t, G)))$$

## Variance

$$\lim_n \frac{1}{n} \text{Var}(N_n(G)) = c(G) \sim 0.56.$$

We prove (non-universality) :

$$\lim_n \frac{1}{n} \text{Var}(N_n(Y)) = c(G) + 30(E(Y_1^4) - 3).$$

Basic quantity

$$\frac{1}{n} \int_0^{\pi n} dt \int_0^{\pi n} ds \quad \partial_1 \Phi_\delta(S_n(t, Y)) \partial_1 \Phi(S_n(s, Y))$$

We use *CLT* for

$$\bar{S}_n(t, s, Y) = (S_n(t, Y), S_n(s, Y)) \in R^4$$

with the Edgorth developpement of order 4.



## Integration by parts

$$F = f(\xi_i V_i + (1 - \xi_i) U_i, i = 1, \dots, n)$$
$$P(V \in dx) = \frac{1}{c_r} \varphi_r(x - x_0) dx \quad \varphi_r \simeq \mathbf{1}_{B_r(0)}$$

Derivatives

$$D_k F = \xi_k \frac{\partial}{\partial V_k} f(\xi_i V_i + (1 - \xi_i) U_i, i = 1, \dots, n)$$

Duality

$$E(\langle DF, DG \rangle) = \sum_{k=1}^n E(D_k F D_k G) = E(FLG)$$

with

$$LG = - \sum_{k=1}^N D_k D_k G + D_k G \partial_v \ln \varphi_r(V_k - x_0)$$

Covariance matrix :  $F = (F_1, \dots, F_d)$

$$\sigma_F^{i,j} = \langle DF_i, DF_j \rangle, \quad i, j = 1, \dots, d.$$

Difficulty

$$P(\det \sigma_F = 0) \geq P(\xi_1 = \dots = \xi_n = 0) > 0.$$

Hoffding's inequality

$$P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i < \frac{c_r}{2}\right) \leq e^{-an}.$$

**Regularization lemma**  $|\alpha| = m$

$$\begin{aligned} |E(\partial^\alpha f(F)) - E(\partial^\alpha (f * \gamma_\varepsilon)(F))| &\leq C \|f\|_{m,\infty} P^{1/2}(\det \sigma_F \leq \eta) + \frac{\varepsilon}{\eta^{2m}} \|f\|_\infty \\ &\leq C \|f\|_{m,\infty} e^{-an} + \frac{\varepsilon}{\eta^{2m}} \|f\|_\infty \end{aligned}$$

Here  $\gamma_\varepsilon$  is a **super kernel**

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2. Bally, V., Caramellino, L., Poly, G. (2017) On the non universality for the variance of the number of real roots of random trigonometric polynomials. Preprint *arXiv* :1711.03364 43 pages. November 2017 *Probability Theory and Related Fields*
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