Geodesics in random geometry

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Random excursions with Jean Bertoin
Outline

The Brownian sphere: Universal model of two-dimensional random geometry that arises as the scaling limit of graphs drawn on the sphere (planar maps) and chosen at random in a suitable class.

→ Similar to the approximation of Brownian motion by rescaled random walks.

Goal of the lecture: Study geodesics in the Brownian sphere.
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gometry that arises as the scaling limit of graphs drawn on the sphere
(planar maps) and chosen at random in a suitable class.

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Goal of the lecture: Study geodesics in the Brownian sphere.

1. The Brownian sphere as the scaling limit of rescaled planar maps.
2. The construction of the Brownian sphere from Brownian motion
   indexed by the Brownian tree.
3. Geodesics from a typical point of the Brownian sphere.
4. Exceptional points: Geodesic stars.
1. The Brownian sphere as the scaling limit of rescaled planar maps

**Definition**

A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere). Loops and multiple edges allowed.
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A rooted triangulation

Faces = connected components of the complement of edges

\( p \)-angulation:

- each face is incident to \( p \) edges

- \( p = 3 \): triangulation

- \( p = 4 \): quadrangulation

Rooted map: distinguished oriented edge
The same planar map:
The same planar map:

Two different planar maps:
A large triangulation of the sphere
Can we get a continuous model out of this?

(Simulation: N. Curien)
Planar maps as metric spaces

*M* planar map

- **\( V(M) = \) set of vertices of *M***
- **\( d_{gr} \) graph distance on \( V(M) \)**
- \( (V(M), d_{gr}) \) is a (finite) **metric space**

In **blue**: distances from the root vertex
Planar maps as metric spaces

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- $V(M) =$ set of vertices of $M$
- $d_{gr}$ graph distance on $V(M)$
- $(V(M), d_{gr})$ is a (finite) metric space

$\mathcal{M}_p^n = \{\text{rooted } p-\text{angulations with } n \text{ faces}\}$

$\mathcal{M}_p^n$ is a finite set (finite number of possible “shapes”)

Choose $M_n$ uniformly at random in $\mathcal{M}_p^n$. We want to let $n \to \infty$ ($p$ fixed)
Planar maps as metric spaces

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Choose \( M_n \) uniformly at random in \( \mathbb{M}_n^p \). We want to let \( n \to \infty \) (\( p \) fixed)

View \( (V(M_n), d_{gr}) \) as a random variable with values in
\[ \mathcal{K} = \{ \text{compact metric spaces, modulo isometries} \} \]
which is equipped with the Gromov-Hausdorff distance.
The Gromov-Hausdorff distance

The Hausdorff distance. \( K_1, K_2 \) compact subsets of a metric space

\[
d_{\text{Haus}}(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}
\]

\((U_\varepsilon(K_1)\) is the \(\varepsilon\)-enlargement of \(K_1\))
The Gromov-Hausdorff distance

The Hausdorff distance. Let $K_1, K_2$ be compact subsets of a metric space.

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Definition (Gromov-Hausdorff distance)

If $(E_1, d_1)$ and $(E_2, d_2)$ are two compact metric spaces, then

\[ d_{\text{GH}}(E_1, E_2) = \inf\{ d_{\text{Haus}}(\psi_1(E_1), \psi_2(E_2)) \} \]

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of $E_1$ and $E_2$ into the same metric space $E$.

\[ \begin{matrix}
E_1 & \psi_1 & E_2 \\
\psi_2 & & \\
\end{matrix} \]
Fact

If $\mathbf{K} = \{ \text{isometry classes of compact metric spaces} \}$, then

$$(\mathbf{K}, d_{GH}) \text{ is a separable complete metric space (Polish space)}$$
Gromov-Hausdorff convergence of rescaled maps

**Fact**

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\((\mathbb{K}, d_{\text{GH}})\) is a separable complete metric space (Polish space)

→ If \( M_n \) is uniformly distributed over \( \{ p - \text{angulations with } n \text{ faces} \} \), it makes sense to study the convergence in distribution as \( n \to \infty \) of

\((V(M_n), n^{-a} d_{\text{gr}})\)

as random variables with values in \( \mathbb{K} \).

(Problem stated for triangulations by O. Schramm [ICM, 2006])
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**Choice of the rescaling factor** \( n^{-a} : \ a > 0 \) is chosen so that
\[
\text{diam}(V(M_n)) \approx n^a.
\]
\[\Rightarrow a = \frac{1}{4} \text{ [cf Chassaing-Schaeffer PTRF 2004 for quadrangulations]}\]
The Brownian sphere

\[ \mathbb{M}^p_n = \{ \text{rooted } p - \text{angulations with } n \text{ faces} \} \]

\( M_n \) uniform over \( \mathbb{M}^p_n \), \( V(M_n) \) vertex set of \( M_n \), \( d_{gr} \) graph distance

Theorem (LG 2013, Miermont 2013)

Suppose that either \( p = 3 \) (triangulations) or \( p \geq 4 \) is even. Set

\[ c_3 = \frac{6}{4}, \quad c_p = \left( \frac{9p(p-2)}{4} \right)^{1/4} \text{ if } p \text{ is even.} \]

Then,

\[ (V(M_n), c_p n^{-1/4} d_{gr}) \to (m_\infty, D) \]

in the Gromov-Hausdorff sense. The limit \( (m_\infty, D) \) is a random compact metric space that does not depend on \( p \) (universality) and is called the Brownian sphere (or Brownian map).

Remarks

• \( p = 3 \) (triangulations) solves Schramm’s problem.
• Extensions to other random planar maps: Abraham, Addario-Berry-Albenque (case of odd \( p \)), Bettinelli-Jacob-Miermont, Marzouk, etc.
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Then,

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Properties of the Brownian sphere

The Brownian sphere is a geodesic space: any pair of points is connected by a (possibly not unique) geodesic. (A Gromov-Hausdorff limit of geodesic spaces is a geodesic space.)
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Theorem (Hausdorff dimension)

\[ \dim(m_\infty, D) = 4 \quad \text{a.s.} \]

(Already “known” in the physics literature.)
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Theorem (topological type)

Almost surely, \((m_{\infty}, D)\) is homeomorphic to the 2-sphere \(S^2\).
The volume measure on the Brownian sphere

Two ways of constructing the volume measure $\text{Vol}(\cdot)$ on $m_\infty$:

- By proving that the approximation by rescaled planar maps holds in the sense of the Gromov-Hausdorff-Prokhorov topology (then $\text{Vol}(\cdot)$ appears as the limit of the uniform probability measure on the vertex set)

- By using the construction of $m_\infty$ as a quotient space of the CRT, and taking the pushforward of the volume measure on the CRT.

Theorem (LG 2021)

Set $h(r) = r^4 \log \log (1/r)$. There exists a constant $c > 0$ such that, a.s. for every Borel subset $A$ of $m_\infty$,

$$h - m(A) = c \text{Vol}(A)$$

where $h - m$ denotes the Hausdorff measure associated with $h$.

Remark. This shows that $\text{Vol}(\cdot)$ is determined by the metric structure.
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- By proving that the approximation by rescaled planar maps holds in the sense of the Gromov-Hausdorff-Prokhorov topology (then \( \text{Vol}(\cdot) \) appears as the limit of the uniform probability measure on the vertex set)

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Connections with Liouville quantum gravity

Miller, Sheffield (2015-2016) have developed a program aiming to relate the Brownian sphere with Liouville quantum gravity:

- new construction of the Brownian sphere using the Gaussian free field and the random growth process called Quantum Loewner Evolution (an analog of the celebrated SLE processes studied by Lawler, Schramm and Werner)
- this construction makes it possible to equip the Brownian sphere with a conformal structure, and in fact to show that this conformal structure is determined by the Brownian sphere.
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More recently: the Miller-Sheffield construction has been simplified by a direct construction of the Liouville quantum gravity metric from the Gaussian free field (Ding-Dubédat-Dunlap-Falconet 2020, uniqueness proved by Gwynne-Miller 2021).
2. The construction of the Brownian sphere

The Brownian sphere \((m_\infty, D)\) is constructed by identifying certain pairs of points in Aldous’ Brownian continuum random tree (CRT).
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Constructions of the CRT (Aldous, 1991-1993):

- As the scaling limit of many classes of discrete trees
- As the random real tree whose contour is a Brownian excursion.
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**Constructions of the CRT (Aldous, 1991-1993):**

- As the scaling limit of many classes of discrete trees
- As the random real tree whose contour is a Brownian excursion.

Coding a (discrete) plane tree by its **contour function** (or Dyck path):
The notion of a real tree

Definition

A real tree, or $\mathbb{R}$-tree, is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique continuous and injective path (up to re-parameterization)
- this path is isometric to a line segment

$\mathcal{T}$ is a rooted real tree if there is a distinguished point $\rho$, called the root.

Remark. A real tree can have infinitely many branching points (uncountably) infinitely many leaves

Fact. The coding of discrete trees by contour functions can be extended to real trees: also gives a cyclic ordering on the tree.
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The real tree coded by a function $g$

$g : [0, 1] \longrightarrow [0, \infty)$
continuous,

$g(0) = g(1) = 0$

$m_g(s, t) = \min_{[s \wedge t, s \vee t]} g$

$g(s)$
$g(t)$
$m_g(s, t)$

$s$
$t$
$t'$

1
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$t \sim t'$ iff $d_g(t, t') = 0$ (or equivalently $g(t) = g(t') = m_g(t, t')$)
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Proposition (Duquesne-LG)

$T_g := [0, 1] / \sim$ equipped with $d_g$ is a real tree, called the tree **coded by** $g$. It is rooted at $\rho = 0$. 
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pseudo-metric on \([0, 1]\)

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**Proposition (Duquesne-LG)**

\[ \mathcal{T}_g := [0, 1]/\sim \text{ equipped with } d_g \text{ is a real tree, called the tree coded by } g. \text{ It is rooted at } \rho = 0. \]

The canonical projection \( p_g : [0, 1] \rightarrow \mathcal{T}_g \) is a cyclic exploration of \( \mathcal{T}_g \)

\[ \text{Makes it possible to define intervals on } \mathcal{T}_g: [a, b] = p_g([s, t]) \text{ where } s \text{ and } t \text{ are such that } p_g(s) = a, p_g(t) = b, [s, t] \text{ as small as possible.} \]
Definition of the CRT

Let \((e_t)_{0 \leq t \leq 1}\) be a Brownian excursion with duration 1.

**Definition**

The CRT \((\mathcal{T}_e, d_e)\) is the (random) real tree coded by the Brownian excursion \(e\).

Simulation of a Brownian excursion
A simulation of the CRT
(simulation: I. Kortchemski)
Assigning Brownian labels to a real tree

Let \((\mathcal{T}, d)\) be a real tree with root \(\rho\).

\((Z_a)_{a \in \mathcal{T}}\): Brownian motion indexed by \((\mathcal{T}, d)\)

= centered Gaussian process such that

- \(Z_\rho = 0\)
- \(E[(Z_a - Z_b)^2] = d(a, b), \quad a, b \in \mathcal{T}\)
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We view \(Z_a\) as a label assigned to \(a \in \mathcal{T}\).

Labels evolve like Brownian motion along the branches of the tree:

- The label \(Z_a\) is the value at time \(d(\rho, a)\) of a standard Brownian motion
- Similar property for \(Z_b\), but one uses
  - the same BM between 0 and \(d(\rho, a \land b)\)
  - an independent BM between \(d(\rho, a \land b)\) and \(d(\rho, b)\)
The definition of the Brownian sphere 
\((\mathcal{T}_e, d_e)\) is the CRT, \((Z_a)_{a \in \mathcal{T}_e}\) Brownian motion indexed by the CRT
(\textbf{Two levels of randomness!}).
The definition of the Brownian sphere $(\mathcal{T}_e, d_e)$ is the CRT, $(Z_a)_{a \in \mathcal{T}_e}$ Brownian motion indexed by the CRT (Two levels of randomness!).

Set, for every $a, b \in \mathcal{T}_e$,

$$D^0(a, b) = Z_a + Z_b - 2 \max \left( \min_{c \in [a, b]} Z_c, \min_{c \in [b, a]} Z_c \right)$$

where $[a, b]$ is the interval from $a$ to $b$ in $\mathcal{T}_e$ (these are the points visited when going from $a$ to $b$ using the cyclic exploration of the tree $\mathcal{T}_e$).
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Then set

\[
D(a, b) = \inf_{a_0=a, a_1, \ldots, a_{k-1}, a_k=b} \sum_{i=1}^{k} D^0(a_{i-1}, a_i),
\]

\(a \approx b\) if and only if \(D(a, b) = 0\) (equivalent to \(D^0(a, b) = 0\)).
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**Definition**

The **Brownian sphere** \(m_\infty\) is the quotient space \(m_\infty := \mathcal{T}_e/\sim\), which is equipped with the distance induced by \(D\).
Summary and interpretation

Starting from the CRT $\mathcal{T}_e$, with Brownian labels $Z_a, a \in \mathcal{T}_e$,

$\rightarrow$ The two vertices $a, b \in \mathcal{T}_e$ are glued ($a \approx b$) if:

- they have the same label $Z_a = Z_b$,
- one can go from $a$ to $b$ around the tree visiting only vertices with label greater than or equal to $Z_a = Z_b$. 

\[ a \approx b \]
A key property of distances in the Brownian sphere

Notation:

- \( \Pi \) is the canonical projection from the CRT \( \mathcal{T}_e \) onto \( m_\infty = \mathcal{T}_e/ \approx \)
- For \( x = \Pi(a) \), \( Z_x := Z_a \) (does not depend on choice of \( a \)).

Fact

Let \( a_\ast \) be the (unique) point of the CRT \( \mathcal{T}_e \) with minimal label, and \( x_\ast = \Pi(a_\ast) \). Then, for every \( x \in m_\infty \),

\[
D(x_\ast, x) = Z_x - \min Z
\]

(“labels” exactly correspond, up to a shift, to distances from \( x_\ast \)).
A key property of distances in the Brownian sphere

Notation:
- $\Pi$ is the canonical projection from the CRT $\mathcal{T}_e$ onto $\mathfrak{m}_\infty = \mathcal{T}_e / \approx$
- For $x = \Pi(a)$, $Z_x := Z_a$ (does not depend on choice of $a$).

Fact

Let $a_*$ be the (unique) point of the CRT $\mathcal{T}_e$ with minimal label, and $x_* = \Pi(a_*)$. Then, for every $x \in \mathfrak{m}_\infty$,

$$D(x_*, x) = Z_x - \min Z$$

(“labels” exactly correspond, up to a shift, to distances from $x_*$).

The Brownian sphere comes with two distinguished points, namely $x_*$ and $x_0 = \Pi(\rho)$ ($\rho$ is the root of $\mathcal{T}_e$)

$\rightarrow$ $x_0$ and $x_*$ are independently uniformly distributed over $\mathfrak{m}_\infty$ (in a sense that can be made precise)

$\rightarrow$ in particular, $x_*$ is a “typical point” of $\mathfrak{m}_\infty$
3. Geodesics from the “typical” point $x_*$

Recall:

- $\Pi : \mathcal{T}_e \to \mathbf{m}_\infty$ canonical projection
- $x_* = \Pi(a_*)$ is such that $Z_{x_*} = \min Z$

Then for every $x \in \mathbf{m}_\infty$,

$$D(x_*, x) = Z_x - \min Z \quad \text{(notation) } \tilde{Z}_x.$$

Let $x = \Pi(a), a \in \mathcal{T}_e$ be any point of $\mathbf{m}_\infty$. Can construct a “simple geodesic” from $x_*$ to $x$ by taking $t_0$ such that $a = p_e(t_0)$ and setting, for $r \in [0, \tilde{Z}_a]$,

$$\varphi_{t_0}(r) = \Pi(p_e(\sup \{t \leq t_0 : \tilde{Z}_{p_e(t)} = r\}))$$

(informally, $= \Pi(\text{last vertex } b \text{ before } a \text{ s.t. } \tilde{Z}_b = r)$)
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Then for every $x \in \mathcal{M}_\infty$,

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Let $x = \Pi(a)$, $a \in \mathcal{T}_e$ be any point of $\mathcal{M}_\infty$. Can construct a “simple geodesic” from $x_*$ to $x$ by taking $t_0$ such that $a = p_e(t_0)$ and setting, for $r \in [0, \tilde{Z}_a]$,

$$\varphi_{t_0}(r) = \Pi\left(p_e\left(\sup\{t \leq t_0 : \tilde{Z}_{p_e(t)} = r\}\right)\right)$$

(informally, $= \Pi$(last vertex $b$ before $a$ s.t. $\tilde{Z}_b = r$))

**Fact**

*All geodesics from $x_*$ are simple geodesics.*
3. Geodesics from the “typical” point $x_*$

Recall:

- $\Pi : \mathcal{T}_e \to \mathcal{M}_\infty$ canonical projection
- $x_* = \Pi(a_*)$ is such that $Z_{x_*} = \min Z$

Then for every $x \in \mathcal{M}_\infty$,

$$D(x_*, x) = Z_x - \min Z = \tilde{Z}_x.$$ 

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**Fact**

All geodesics from $x_*$ are simple geodesics.

**Remark.** If $a$ is not a leaf, there are several possible choices of $t_0$, corresponding to which side of $a$ one starts from.
The main result about geodesics to a typical point
Define the skeleton of $T_e$ by $Sk(T_e) = T_e \{\text{leaves of } T_e\}$ and set $Skel = \Pi(Sk(T_e))$, where $\Pi : T_e \to T_e/\approx = m_\infty$ canonical projection.
Then
- the restriction of $\Pi$ to $Sk(T_e)$ is a homeomorphism onto $Skel$
- $\dim(Skel) = 2$ (recall $\dim(m_\infty) = 4$)
The main result about geodesics to a typical point

Define the skeleton of $T_e$ by $Sk(T_e) = T_e \setminus \{\text{leaves of } T_e\}$ and set $\text{Skel} = \Pi(Sk(T_e))$, where $\Pi : T_e \to T_e/\sim = m_\infty$ canonical projection.

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Theorem (Geodesics from the root)

Let $x \in m_\infty$. Then,

- if $x \notin \text{Skel}$, there is a unique geodesic from $x_*$ to $x$
- if $x \in \text{Skel}$, the number of distinct geodesics from $x_*$ to $x$ is the multiplicity $m(x)$ of $x$ in $\text{Skel}$ (note: $m(x) \leq 3$).
The main result about geodesics to a typical point

Define the skeleton of $\mathcal{T}_e$ by $\text{Sk}(\mathcal{T}_e) = \mathcal{T}_e \setminus \{\text{leaves of } \mathcal{T}_e\}$ and set $\text{Skel} = \Pi(\text{Sk}(\mathcal{T}_e))$, where $\Pi : \mathcal{T}_e \to \mathcal{T}_e / \approx = \mathfrak{m}_\infty$ canonical projection.

Then

- the restriction of $\Pi$ to $\text{Sk}(\mathcal{T}_e)$ is a homeomorphism onto $\text{Skel}$
- $\dim(\text{Skel}) = 2$ (recall $\dim(\mathfrak{m}_\infty) = 4$)

**Theorem (Geodesics from the root)**

*Let $x \in \mathfrak{m}_\infty$. Then,*

- if $x \notin \text{Skel}$, there is a unique geodesic from $x_*$ to $x$
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**Remarks**

- $\text{Skel}$ is the cut-locus of $\mathfrak{m}_\infty$ relative to $x_*$: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if $x_*$ replaced by a point chosen “at random” in $\mathfrak{m}_\infty$.
Illustration of the cut-locus

The cut-locus $\text{Skel}$ is homeomorphic to a non-compact real tree and is dense in $m_{\infty}$.

Geodesics to $x_*$ do not visit Skel (except possibly at their starting point) but “move around” Skel.
Confluence property of geodesics

Proposition

Given $\delta > 0$, there exists $\varepsilon > 0$ s.t.

- if $D(x_*, x) \geq \delta$, $D(x_*, y) \geq \delta$
- if $\gamma$ is any geodesic from $x_*$ to $x$
- if $\gamma'$ is any geodesic from $x_*$ to $y$

then

$\gamma(t) = \gamma'(t)$ for all $t \leq \varepsilon$

“Only one way” of leaving $x_*$ along a geodesic.

(more generally true for a typical point of $m_\infty$)
Why the confluence property

Let $a, b \in \mathcal{T}_e$ such that $a_* \notin [a, b]$ (otherwise interchange $a$ and $b$). Suppose (for simplicity) that $a$ and $b$ are leaves and let $\varphi_a$ and $\varphi_b$ be the simple geodesics from $x_*$ to $\Pi(a)$ and $\Pi(b)$ respectively. Then

$$\varphi_a(t) = \varphi_b(t) \text{ for every } 0 \leq t \leq \min_{d \in [a, b]} Z_d - \min Z(> 0)$$
4. Geodesics from exceptional points

If $x$ and $y$ are typical points of the Brownian sphere (chosen according to the volume measure $\text{Vol}(\cdot)$),

- There is a unique geodesic from $y$ to $x$ (can take $x = x_*$, the cut-locus has zero volume)
4. Geodesics from exceptional points

If $x$ and $y$ are typical points of the Brownian sphere (chosen according to the volume measure $\text{Vol}(\cdot)$)

- There is a **unique geodesic** from $y$ to $x$ (can take $x = x_*$, the cut-locus has zero volume)

If $x$ is typical (say $x = x_*$) then for “exceptional points” $y$:

- There can be **up to 3 geodesics** from $y$ to $x$. 

4. Geodesics from exceptional points

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If $x$ is typical (say $x = x_*$) then for “exceptional points” $y$:

- There can be **up to 3 geodesics** from $y$ to $x$.

If $x$ and $y$ are both exceptional:

- There can be **up to 9 geodesics** from $y$ to $x$ (Miller-Qian (2020), following earlier work of Angel, Kolesnik, Miermont (2017)).
- **Miller-Qian (2020)** even compute the Hausdorff dimension of the set of pairs $(x, y)$ such that there are exactly $k$ geodesics from $y$ to $x$. 
Geodesic stars

A point $x$ of the Brownian sphere $\mathbb{m}_\infty$ is called a geodesic star with $n$ arms ($n \geq 2$) if it is the endpoint of $n$ geodesics that are disjoint (except for their terminal point).
Geodesic stars

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A typical point is not a geodesic star (because of the confluence property!)

A geodesic star with 4 arms
Geodesic stars

A point \( x \) of the Brownian sphere \( \mathbb{m}_\infty \) is called a **geodesic star** with \( n \) arms \((n \geq 2)\) if it is the endpoint of \( n \) geodesics that are disjoint (except for their terminal point).

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**Theorem (Miller-Qian 2020, LG 2021)**

Let \( S_n \) be the set of all geodesic stars with \( n \) arms. Then, for \( n = 2, 3, 4, \)

\[ \dim(S_n) = 5 - n \]
### Geodesic stars

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#### Theorem (Miller-Qian 2020, LG 2021)

Let $S_n$ be the set of all geodesic stars with $n$ arms. Then, for $n = 2, 3, 4$,

$$\dim(S_n) = 5 - n$$

**Open problem**: Is $S_5$ not empty?

**Remark**. Miller and Qian proved that the set of all interior points of geodesics has dimension 1. An interior point of a geodesic is a geodesic star with 2 arms, but not a typical one!
Results on geodesics in the Brownian sphere

- **Miermont 2009** (Ann. ENS): uniqueness of the geodesic between two typical points (also in higher genus)

- **LG 2010** (Acta Math.): complete description of geodesics from a typical point

- **Miermont 2013** (Acta Math.): uses the notion of geodesic stars to prove uniqueness of the Brownian sphere

- **Angel, Kolesnik, Miermont 2017** (Ann. Probab.): first results about geodesic networks (union of the geodesics connecting two points)

- **Miller, Qian 2020**: full description of geodesic networks. Upper bound on the dimension of geodesic stars

- **LG 2021**: lower bound matching the upper bound of Miller-Qian for geodesic stars.

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Sketch of proof of the lower bound $\dim(S_n) \geq 5 - n$

From now on, consider the “free Brownian sphere” (with a random volume) under the $\sigma$-finite measure $\mathbb{N}_0$.

Define a notion of $\varepsilon$-approximate geodesic stars: for $\varepsilon > 0$, $x \in \mathbf{m}_\infty$ belongs to $S^n_\varepsilon$ if there are $n$ geodesics to $x$ starting from the boundary of the ball of radius 1 centered at $x$ that are disjoint up to the time when they arrive at distance $\varepsilon$ from $x$. 

Then, if $\text{Vol} \left( S^n_\varepsilon \right)$ is the volume measure on $\mathbf{m}_\infty$, for $n = 2, 3, 4$, 

$$
\mathbb{E}_{\mathbb{N}_0} \left( \text{Vol} \left( S^n_\varepsilon \right) \right) \geq c \varepsilon^{n-1}
$$

and for every $\delta > 0$, 

$$
\mathbb{E}_{\mathbb{N}_0} \left( \int \int 1_{S^n_\varepsilon \times S^n_\varepsilon} (x, y) \, d\text{Vol}(x) \, d\text{Vol}(y) - (5 - n - \delta) \text{Vol}(d) \right) \leq c \delta^{\varepsilon^2} (n-1)
$$

$\rightarrow$ Standard techniques (extraction of convergent subsequence from $\varepsilon - (n-1) \text{Vol}|_{S^n_\varepsilon}$, Frostman lemma) then show that $S^n = \bigcap_{\varepsilon > 0} S^n_\varepsilon$ has dimension $\geq 5 - n$ on an event of positive $\mathbb{N}_0$-measure.
Sketch of proof of the lower bound \( \dim(S_n) \geq 5 - n \)

From now on, consider the “free Brownian sphere” (with a random volume) under the \( \sigma \)-finite measure \( \mathbb{N}_0 \).

Define a notion of \( \varepsilon \)-approximate geodesic stars: for \( \varepsilon > 0 \), \( x \in \mathbb{m}_\infty \) belongs to \( S_{\varepsilon n} \) if there are \( n \) geodesics to \( x \) starting from the boundary of the ball of radius 1 centered at \( x \) that are disjoint up to the time when they arrive at distance \( \varepsilon \) from \( x \).

Then, if \( \text{Vol}(\cdot) \) is the volume measure on \( \mathbb{m}_\infty \), for \( n = 2, 3, 4 \),

\[
\mathbb{E}_{\mathbb{N}_0} \left( \text{Vol}(S_{\varepsilon n}) \right) \geq c \varepsilon^{n-1}
\]

and for every \( \delta > 0 \),

\[
\mathbb{E}_{\mathbb{N}_0} \left( \iint 1_{S_{\varepsilon n} \times S_{\varepsilon n}}(x, y) D(x, y)^{-(5-n-\delta)} \text{Vol}(dx)\text{Vol}(dy) \right) \leq c_\delta \varepsilon^{2(n-1)}
\]
Sketch of proof of the lower bound $\dim(S_n) \geq 5 - n$

From now on, consider the “free Brownian sphere” (with a random volume) under the $\sigma$-finite measure $\mathbb{N}_0$.

Define a notion of $\varepsilon$-approximate geodesic stars: for $\varepsilon > 0$, $x \in m_\infty$ belongs to $S_\varepsilon^n$ if there are $n$ geodesics to $x$ starting from the boundary of the ball of radius 1 centered at $x$ that are disjoint up to the time when they arrive at distance $\varepsilon$ from $x$.

Then, if $\text{Vol}(\cdot)$ is the volume measure on $m_\infty$, for $n = 2, 3, 4$,

$$\mathbb{E}_{\mathbb{N}_0}(\text{Vol}(S_\varepsilon^n)) \geq c\varepsilon^{n-1}$$

and for every $\delta > 0$,

$$\mathbb{E}_{\mathbb{N}_0}\left(\int\int 1_{S_\varepsilon^n \times S_\varepsilon^n}(x, y) D(x, y)^{-(5-n-\delta)} \text{Vol}(dx)\text{Vol}(dy)\right) \leq c_\delta \varepsilon^{2(n-1)}$$

$\rightarrow$ Standard techniques (extraction of convergent subsequence from $\varepsilon^{-(n-1)}\text{Vol}|_{S_\varepsilon^n}$, Frostman lemma) then show that $S_n = \bigcap_{\varepsilon > 0} S_\varepsilon^n$ has dimension $\geq 5 - n$ on an event of positive $\mathbb{N}_0$-measure.
A useful tool: hulls

Let $x, y \in m_\infty$ and $r > 0$. Write $B_r(x)$ for the closed ball of radius $r$ centered at $x$. On the event $\{D(x, y) > r\}$, one can define the hull of radius $r$ centered at $x$ relative to $y$, denoted by $B_{r}^{\bullet, y}(x)$:
A useful tool: hulls

Let $x, y \in m_\infty$ and $r > 0$. Write $B_r(x)$ for the closed ball of radius $r$ centered at $x$. On the event $\{D(x, y) > r\}$, one can define the hull of radius $r$ centered at $x$ relative to $y$, denoted by $B_{r, y}^\circ(x)$:

**Definition**

The complement of $B_{r, y}^\circ(x)$ is the connected component of the complement of $B_r(x)$ that contains $y$. 

Jean-François Le Gall  (Univ. Paris-Saclay)  
Geodesics in random geometry  
Jean Bertoin’s conference
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Cactus representation of the Brownian sphere (the vertical coordinate here is the distance from $x$)
A useful tool: hulls

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**Definition**

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Cactus representation of the Brownian sphere (the vertical coordinate here is the distance from $x$)

One can make sense of the boundary size $|\partial B_{r, y}(x)|$ of the hull (at least when $x$ and $y$ are “typical”)

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Forest representation of a hull

Consider the hull \( B_r^{x_0}(x_\ast) \) and its boundary size \( |\partial B_r^{x_0}(x_\ast)| \).

Then conditionally on \( |\partial B_r^{x_0}(x_\ast)| = u \), one can represent the hull in terms of a Poisson forest of real trees equipped with Brownian labels:

\[
(T_{e_i}, (Z^i_a)_{a \in T_{e_i}}), \quad i \in I
\]

where \( \sum_{i \in I} \delta_{e_i} \) is Poisson with intensity \( u \, n(de) \) (here \( n \) is the Itô excursion measure). Consider the trees as planted uniformly over \([0, u]\), and identify 0 with \( u \). Furthermore condition the minimal label of the forest to be equal to \(-r\).
Forest representation of a hull

Consider the hull $B_{r,x_0}^\bullet(x_*)$ and its boundary size $|\partial B_{r,x_0}^\bullet(x_*)|$. Then conditionally on $|\partial B_{r,x_0}^\bullet(x_*)| = u$, one can represent the hull in terms of a Poisson forest of real trees equipped with Brownian labels:

$$(\mathcal{T}_{e_i}, (Z^i_a)_{a \in \mathcal{T}_{e_i}}), \quad i \in I$$

where $\sum_{i \in I} \delta_{e_i}$ is Poisson with intensity $u n(de)$ (here $n$ is the Itô excursion measure). Consider the trees as planted uniformly over $[0, u]$, and identify $0$ with $u$. Furthermore condition the minimal label of the forest to be equal to $-r$.

Then the hull $B_{r,x_0}^\bullet(x_*)$ equipped with its intrinsic distance is obtained from the labeled forest by exactly the same construction as the Brownian sphere from $(\mathcal{T}_e, (Z_a)_{a \in \mathcal{T}_e})$.

Labels shifted by $+r$ again correspond to distances from the point $x_*$, which is the point with minimal label.
The one-point estimate

The hull $B_{r, x_0}^*(x_*)$ is obtained from the labeled forest by exactly the same construction as the Brownian sphere from $(\mathcal{T}_e, (Z_a)_{a \in \mathcal{T}_e})$.

Labels shifted by $+r$ correspond to distances from the point $x_*$, which is the point with minimal label.
The one-point estimate

The hull $B_{r,x_0}(x_*)$ is obtained from the labeled forest by exactly the same construction as the Brownian sphere from $(\mathcal{T}_e, (Z_a)_{a \in \mathcal{T}_e})$.

Labels shifted by $+r$ correspond to distances from the point $x_*$, which is the point with minimal label.

In particular, geodesics to $x_*$ are constructed in the same manner (going backward in the forest in order to meet points with smaller and smaller label until reaching $x_*$):
The one-point estimate

The hull $B_{r,x_0}^\cdot(x_\ast)$ is obtained from the labeled forest by exactly the same construction as the Brownian sphere from $(T_e, (Z_a)_{a \in T_e})$.

Labels shifted by $+r$ correspond to distances from the point $x_\ast$, which is the point with minimal label.

In particular, geodesics to $x_\ast$ are constructed in the same manner (going backward in the forest in order to meet points with smaller and smaller label until reaching $x_\ast$):

$\longrightarrow$ The event that $x_\ast$ is an $\varepsilon$-approximate geodesic star with $n$ arms occurs if and only if in addition to the tree carrying $x_\ast$ there are $n - 1$ trees in the forest carrying vertices with label $<-r+\varepsilon$.

$\longrightarrow$ The probability of this event is $\approx \varepsilon^{n-1}$. 
An ingredient for the two-point estimate

Recall that $x_*$ and $x_0$ are the two distinguished points of $\mathbf{m}_\infty$ (distributed independently and uniformly).

**Theorem**

Let $r > 0$. Conditionally on the event $\{D(x_*, x_0) > 2r\}$, the hulls $B_{r,x_0}(x_*)$ and $B_{r,x_*}(x_0)$ viewed as (measure) metric spaces for their intrinsic distances are independent conditionally on their boundary sizes, and their conditional distribution can be described as before from a Poisson labeled forest.

This is a kind of spatial Markov property of the Brownian sphere (valid only for the free Brownian sphere!).

Related results: Riera’s thesis (2021)
Joyeux anniversaire Jean !