Branching-invariant point measures

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Random excursions with Jean Bertoin

Introduction: Lévy processes

Definition

A Lévy process is a right-continuous process X such that

- $X_0 = 0$ a.s.
- for all $s, t \geq 0$, $X_{t+s} = X_t + \overline{X}_s$, with $\overline{X}_s$ an independent copy of X_s.

Lévy processes are the natural continuous-time extensions, and typical scaling limits, of random walks. Indeed

$$(X_t, t \geq 0) \text{ Lévy process } \iff \forall t > 0, \ (X_{tn}, n \in \mathbb{N}) \text{ random walk.}$$

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Introduction: Lévy processes

Some classical results

Lévy-Khinchine formula

There exists a one-to-one map between Lévy processes and triplets $\sigma^2 \geq 0$, $a \in \mathbb{R}$ and π measure on $\mathbb{R}^*$ s.t. $\int 1 \wedge x^2 \pi(\mathrm{d}x) < \infty$, via

$$\mathbf{E}\left(e^{i\xi X_t}\right) = \exp\left[t\left(-\frac{\sigma^2}{2}\xi^2 + ia\xi + \int_{\mathbb{R}} e^{i\xi x} - 1 - i\xi x \mathbf{1}_{\{|x|<1\}}\pi(dx)\right)\right]$$

Lévy-Itô decomposition

One can construct the Lévy process associated to (σ^2, a, π) via

$$X_t = \sigma^2 B_t + at + \int_{[0,t]\times\{|x|\geq 1\}} xN(\mathrm{d}t\mathrm{d}x) + \int_{[0,t]\times\{|x|\geq 1\}} xN^c(\mathrm{d}t\mathrm{d}x)$$

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Classes of E-valued right-continuous processes $(X_t, t \geq 0)$ such that $X_{t+s} \overset{(d)}{=} X_t \boxplus \overline{X}_s$ appear in many contexts :

- **Exchangeable coalescent processes**
- **Fragmentation processes**
- **Branching processes**

Introduction: Lévy processes

Lévy extensions in various spaces

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Outline

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1. Infinitely ramified point measures and branching Lévy processes

2. Stable branching Lévy processes

3. Fixed points of the branching convolution equation

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Infinitely ramified point measures

Random point measures

Definition

Denote by $\mathfrak{P}$ the set of random point measures on $\mathbb{R}$ giving finite mass to $\mathbb{R}_+$, which is identified with the set of decreasing sequences converging to $-\infty$ via

$$D \in \mathfrak{P} \iff D = \sum_{j=1}^{+\infty} \delta_{x_j}, \text{ with } x_1 \geq x_2 \geq \ldots \text{ and } \lim_{j \to +\infty} x_j = -\infty.$$

Notation

For all $x \in \mathbb{R}$ and $D \in \mathfrak{P}$, we set $\tau_x D = \sum_{j \in \mathbb{N}} \delta_{x+x_j}$ the translation operator.

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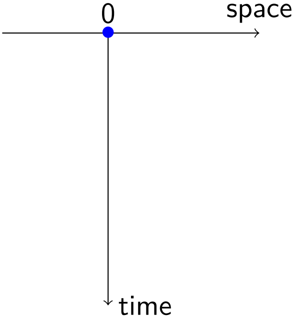
Infinitely ramified point measures

Branching random walk



Description

- An individual is alive at time 0.
- Gives birth to children around its current position.
- Each child reproduces independently with the same law.
- Each new generation reproduces independently.

Notation

We set Z_n the point measure associated to the positions of individuals at time n.

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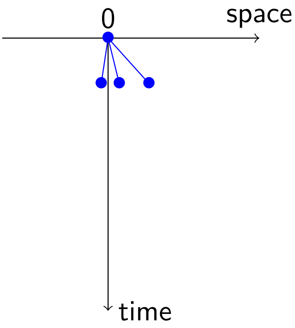

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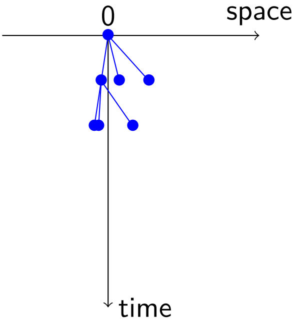

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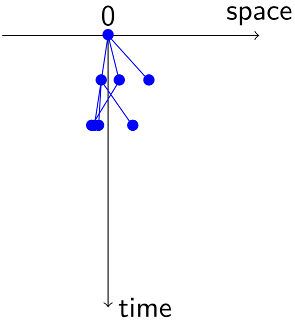

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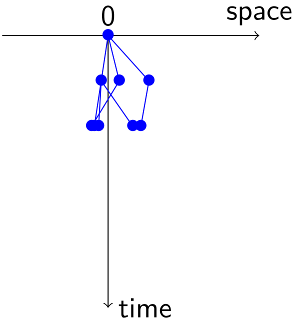

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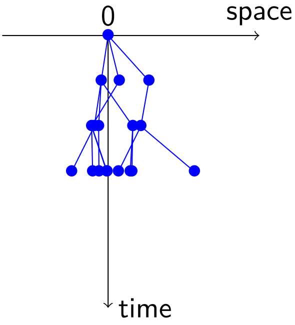

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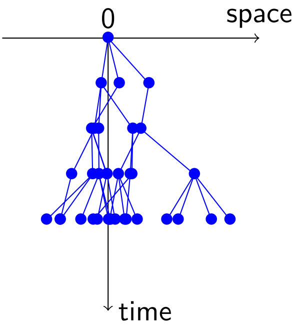

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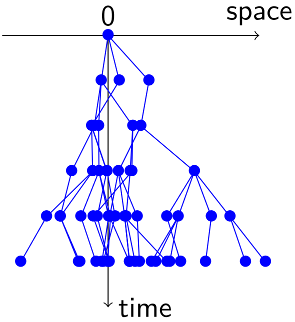

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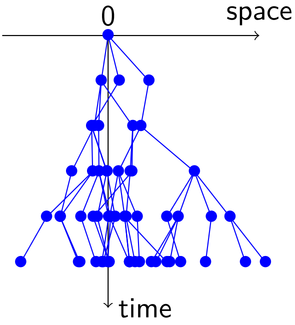

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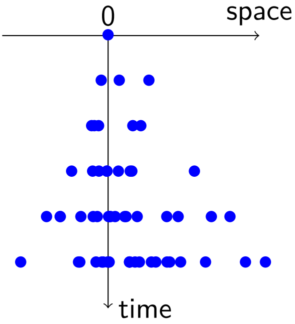

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Infinitely ramified point measures

Definition of the branching random walk

Definition

Let $(D_{n,j}, n, j \in \mathbb{N})$ be a family of i.i.d. random variables in $\mathfrak{P}$. We assume there exists $\theta \geq 0$ satisfying

$$\mathbf{E}\left(\int_{\mathbb{R}} e^{\theta x} D_{1,1}(dx) \right) = \mathbf{E}\left(\sum_{j=1}^{\infty} e^{\theta d_{1,1,j}} \right) < +\infty.$$

A branching random walk with reproduction law $\mu_{1,1}$ is a process $(Z_n, n \in \mathbb{N})$ taking values in the space $\mathfrak{P}$, defined via the formula

$$Z_0 = \delta_0 \quad \text{and} \quad Z_{n+1} = \sum_{j \in \mathbb{N}} \tau_{z_{n,j}} \mu_{(n+1),j},$$

where $(z_{n,j}, j \in \mathbb{N})$ is the sequence of atoms associated to the point measure Z_n.

Infinitely ramified point measures

The branching-convolution operation

Definition

Let $\mathcal{D}$ and $\mathcal{E}$ be the laws of two point measures. Denote by $\mathcal{D} \circledast \mathcal{E}$ the law obtained by making one step of branching random walk with the law $\mathcal{E}$, then one step of branching random walk with the law $\mathcal{E}$.

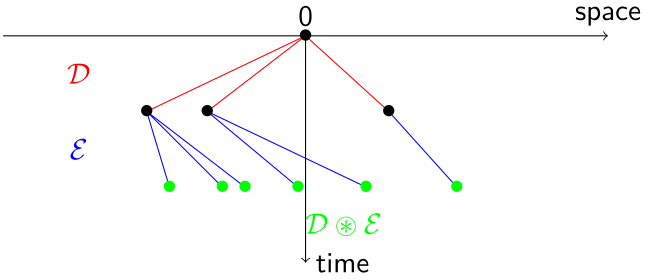

Figure: Branching convolution operator

Infinitely ramified point measures

The branching convolution operation

Remark: link with the convolution operation

Writing $(Z_n, n \geq 0)$ a branching random walk and $\mathcal{Z}$ the law of Z_1, for all $n \in \mathbb{N}$, the law of Z_n is

$$\mathcal{Z}_n = \mathcal{Z}_{n-1} \circledast \mathcal{Z} = \mathcal{Z} \circledast \mathcal{Z}_{n-1} = \mathcal{Z}^{\circledast n}.$$

Remark: continuity

The branching convolution operator is not continuous for the topology of vague convergence of point measures.
If $\mathcal{R}_n$ is the law of $n\delta_{-n}$ and $\mathcal{Y}_n(\{\delta_n\}) = 1 - \mathcal{Y}_n(\{0\}) = 1/n$, then $\mathcal{R}_n \circledast \mathcal{Y}_n$ converges to $N\delta_0$, with N a Poisson random variable.'

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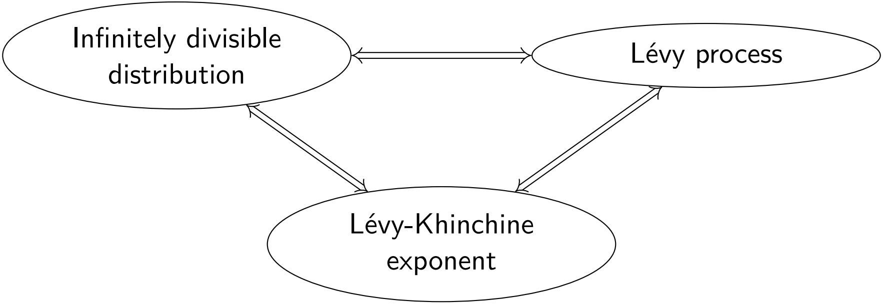
Infinitely ramified point measures

Three classes in correspondence



Infinitely ramified point measures

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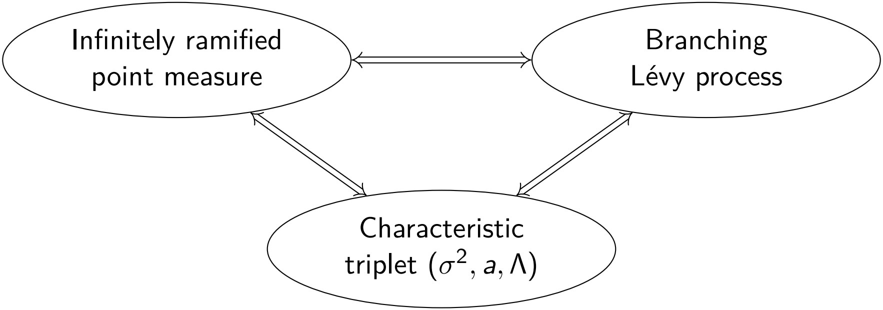

Infinitely ramified point measures

Definition of the IRPM

Definition

The law $\mathcal{Z}$ of a point measure is infinitely ramified if for all $n \in \mathbb{N}$, there exists a branching random walk $Z^{(n)}$ such that $Z^{(n)}_n$ has law $\mathcal{Z}$.

In other words, for all $n \in \mathbb{N}$, there exists a probability distribution $\mathcal{D}_n$ on $\mathfrak{P}$ such that $\mathcal{Z} = \mathcal{D}_n^{\circledast n}$.

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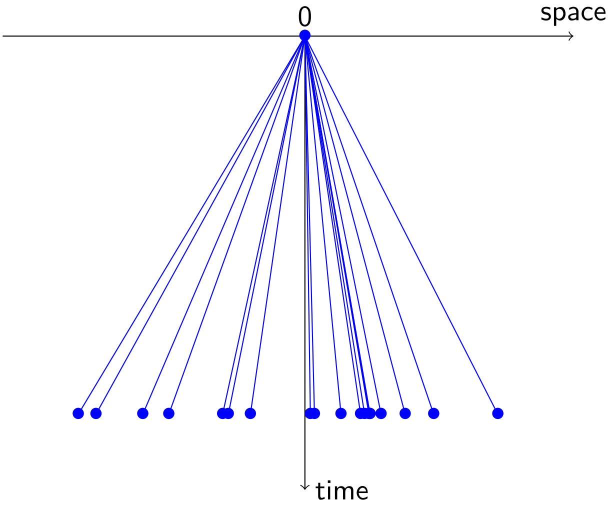
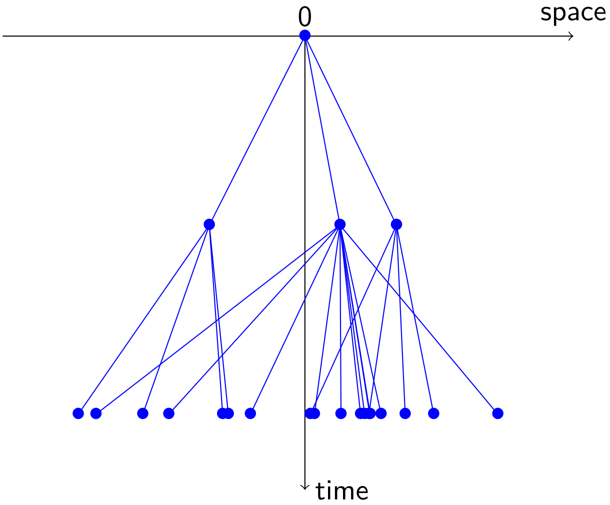


Figure: Some decompositions of an infinitely ramified point measure



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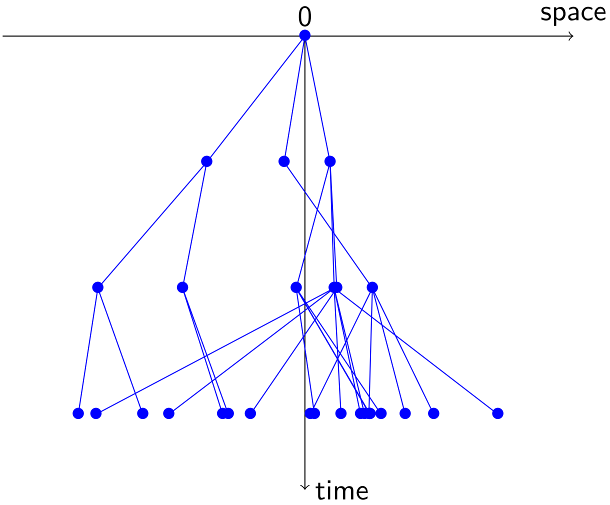

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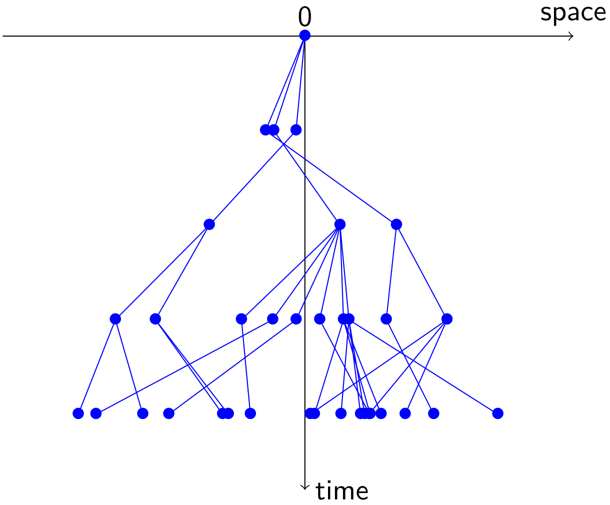

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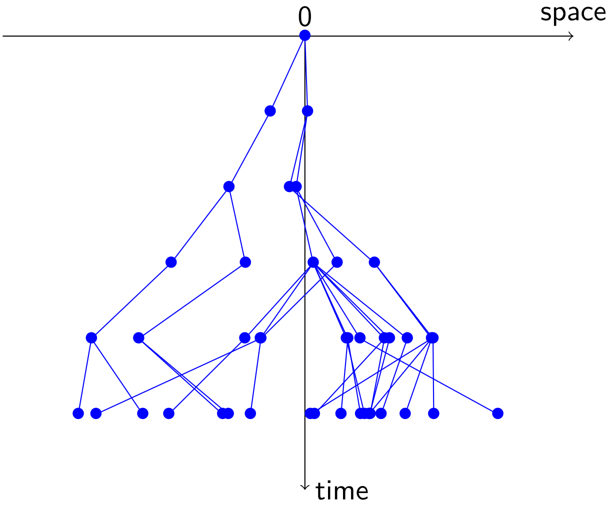

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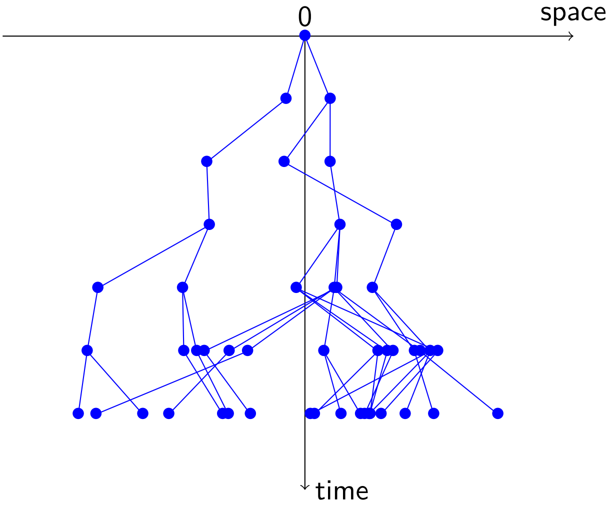

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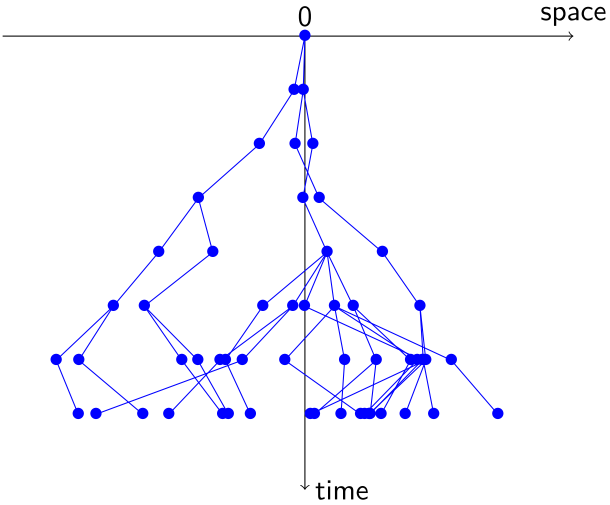

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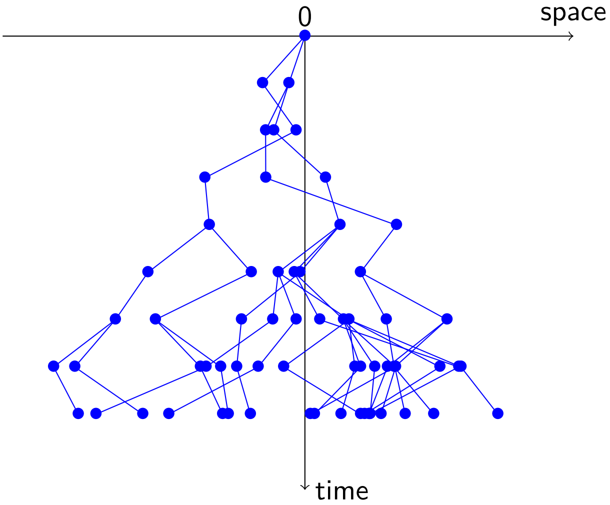

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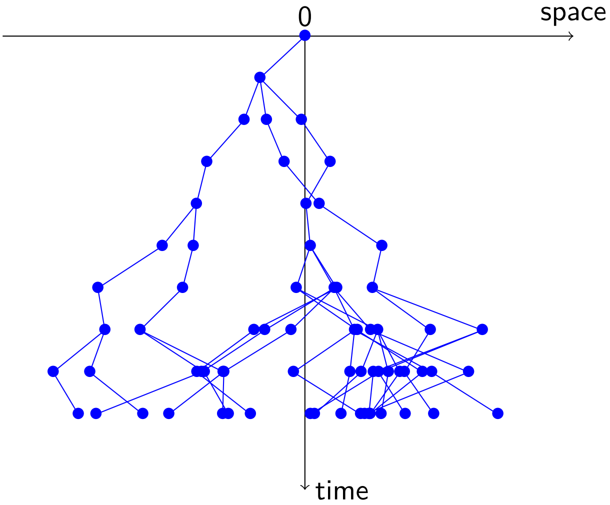

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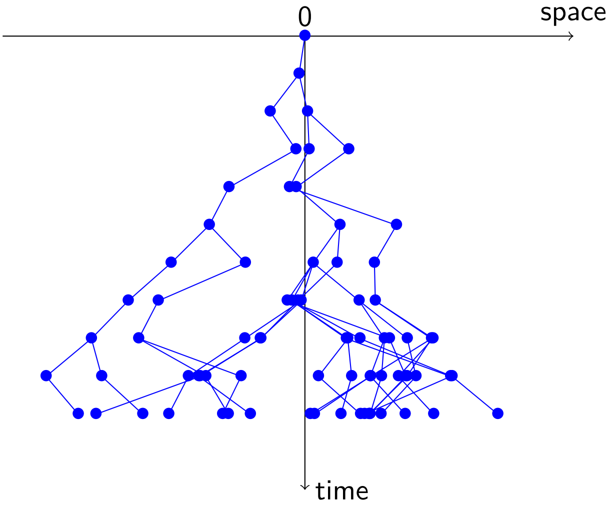

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Branching Lévy processes

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A branching Lévy process is a $\mathfrak{P}$-valued right-continuous process $(Z_t, t \geq 0)$ such that for all $t > 0$, $(Z_{nt}, n \geq 0)$ is a branching random walk.

Remark

A process $(Z_t, t \geq 0)$ is a branching Lévy process if it is right-continuous and satisfies the branching property, i.e. for all $s, t \geq 0$,

$$Z_{t+s} \overset{(d)}{=} \sum_{j \in \mathbb{N}} \tau_{z_j} \overline{Z}_s^{(j)},$$

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Branching Lévy processes

Main results

Theorem (Bertoin, M. (2018))

Let Z be a random point measure with infinitely ramified distribution. We assume there exists $\theta \geq 0$ such that $\mathbf{E}(\int e^{\theta x} Z(dx)) < \infty$. There exists a branching Lévy process $(Z_t, t \in \mathbb{R}_+)$ such that $Z_1 = Z$ in law.

Theorem (Bertoin, M. (2018))

The law of a branching Lévy process satisfying $\mathbf{E}(\int e^{\theta x} Z_1(\mathrm{d}x)) < \infty$ is characterized by a triplet (σ^2, a, Λ) satisfying

$$\sigma^2 \geq 0, \ \ a \in \mathbb{R}, \ \ \int (1 \wedge x_1^2) + \sum_{k=1}^{+\infty} e^{\theta x_k} - 1 - \theta x_1 \mathbf{1}_{\{|x_1| \leq 1\}} \Lambda(d\mathbf{x}) < \infty,$$

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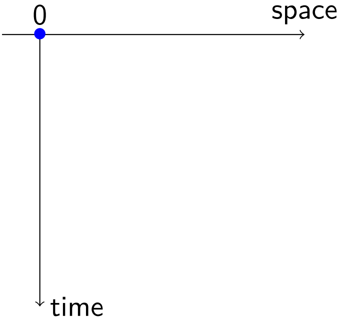
Branching Levy process

Construction for finite birth intensity

We assume that $\int_{\mathfrak{P}} (D(\mathbb{R}) - 1)\Lambda(\mathrm{d}D) < +\infty$.



Description

- An individual alive at time 0.
- Moves according to a Lévy process during its lifetime, an independent exponential variable.
- Makes children around its position according to a point measure at its time of death.
- Each child then starts an independent process.

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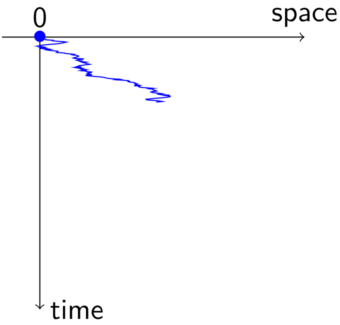

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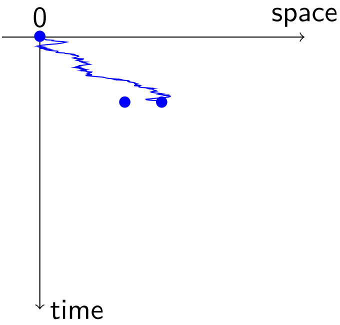

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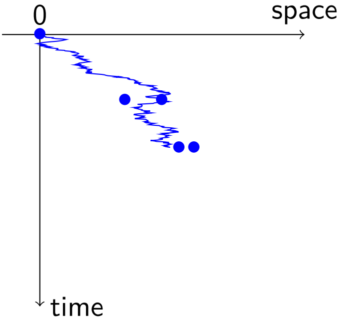
Branching Levy process

Construction for finite birth intensity

We assume that $\int_{\mathfrak{P}} (D(\mathbb{R}) - 1)\Lambda(\mathrm{d}D) < +\infty$.



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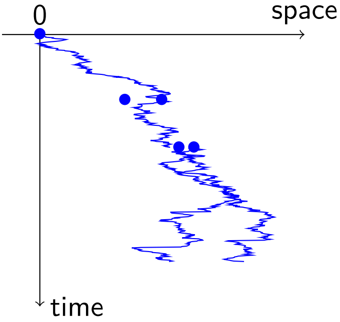

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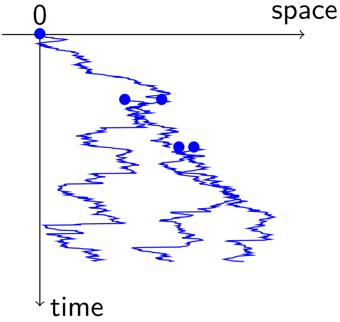

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Construction for finite birth intensity

We assume that $\int_{\mathfrak{P}}(\mu(\mathbb{R})-1)d\Lambda < +\infty$.

Notation

Let (σ^2, a, Λ) be the parameters of the branching Lévy process

- Each individual moves according to a Lévy process with characteristics (σ^2, a, π) with

$$\pi(A) = \Lambda(x_1 \in A, x_2 = -\infty).$$

- The branching rate is given by $\lambda = \Lambda(x_1 = -\infty \text{ ou } x_2 > -\infty)$.
- The offspring creation law is given by $\Lambda_{|\{x_1=-\infty \text{ ou } x_2>-\infty\}}/\lambda$.

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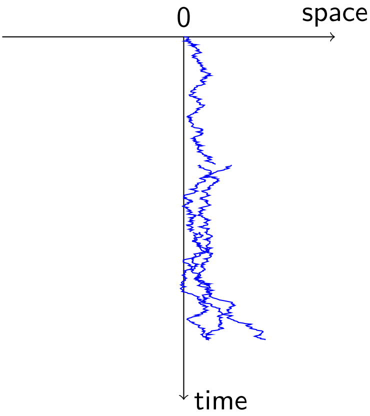
Infinite birth intensity

For all k, we denote by $\Lambda^{(k)}$ the image measure obtained by erasing atoms in $(-\infty, -k]$, which has finite birth intensity.



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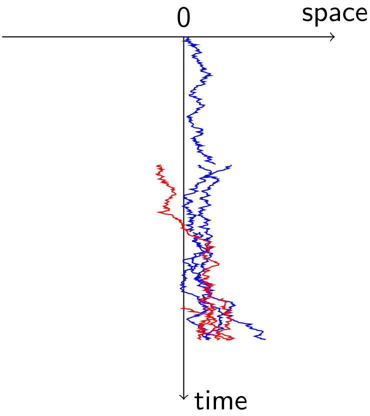

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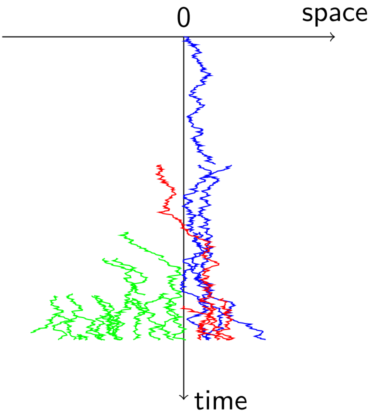

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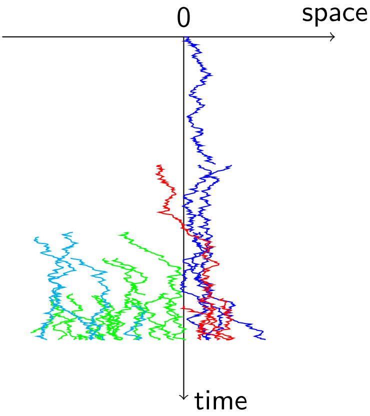

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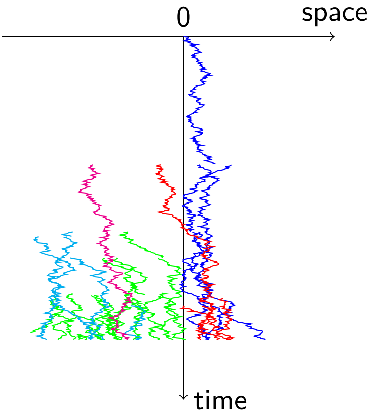

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Examples

1. The branching Lévy process with characteristics $(\sigma^2, 0, \beta\delta_{0,0,-\infty,\ldots})$ is the branching Brownian motion with diffusion coefficient σ^2 and branching rate β.
2. The branching Lévy process with characteristics $(0, 0, \Lambda)$ with Λ a probability measure on $\mathfrak{P}$ is a continuous-time branching random walk in which every individual survives for an exponential random time with parameter 1 before giving birth to children that are positioned around their parent according to the law Λ.
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Plan

Stable branching Lévy processes

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$$\lambda D = (\lambda d_j, j \in \mathbb{N})$$

the point measure obtained by the dilatation by λ of all atoms in the point measure D.

Stable branching Lévy process

A branching Lévy process $(Z_t, t \geq 0)$ is called stable if for all $\lambda > 0$ and $t > 0$,

$$(Z_{\lambda t}, t \geq 0) \overset{(d)}{=} (c(\lambda) Z_t, t \geq 0).$$

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Exponent of a stable branching Lévy process

Barring degenerate cases (empty process and process with a unique static particle), $c(\lambda)$ is uniquely defined for all $\lambda > 0$. In particular, one has $c(\lambda\mu) = c(\lambda)c(\mu)$.

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Characterization of stable branching Lévy processes

Theorem (Bertoin, Cortines, M. (2018))

1. *If $\alpha > 2$, there is no non-degenerate α-stable branching Lévy process.*
2. *If $0 < \alpha \leq 2$, non-degenerate α-stable branching Lévy processes are α-stable Lévy processes (no branching can occur).*
3. *There is no non-degenerate 0-stable branching Lévy process. If $\alpha < 0$, a non-degenerate α-stable branching Lévy process has characteristics $(0, 0, \Lambda)$, with Λ a point measure satisfying*

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Stable branching Lévy processes

Illustration of a process with negative exponent

In other words, a stable branching Lévy process with negative exponent can be thought of as a Crump-Mode-Jagers-type process, in which particles never move or die out, but give birth at rate $y^{-\alpha-1}\mathrm{d}y\mathcal{R}(\mathrm{d}D)$ to children distributed according to the point measure D.

Figure: Birth times and positions of particles in a (-1)-stable branching random walk constructed by successive generations of a CMJ process

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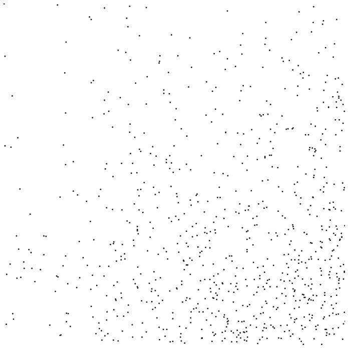

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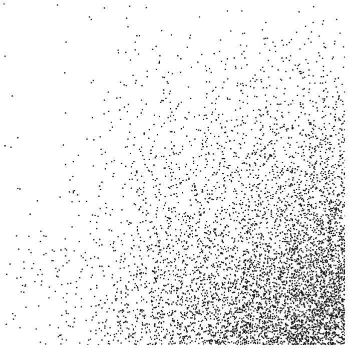

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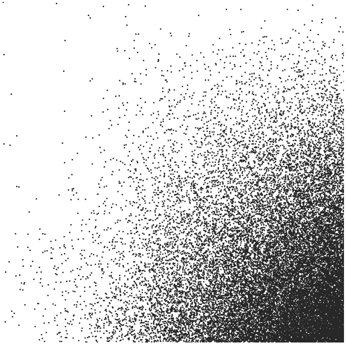

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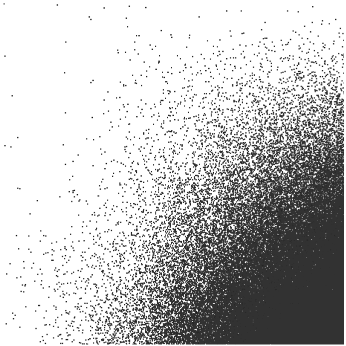

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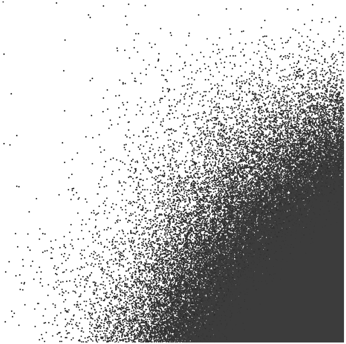

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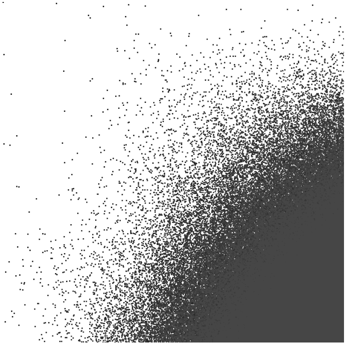

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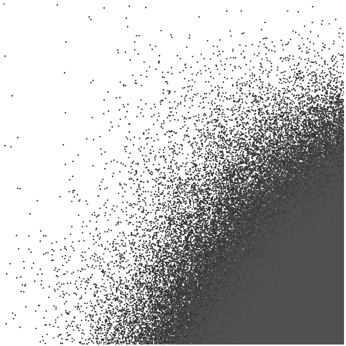

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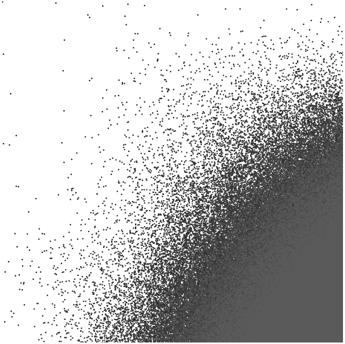

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Theorem (Bertoin, Yang (2021+))

Let $(Z_n, n \geq 0)$ be a branching random walk such that:

1. *The largest atom of Z_1 is 0 a.s.*
2. *Writing x_2 the second largest atom of Z_1, $\mathbf{P}(x_2 \geq -t)$ is regularly varying at $0+$ with index $-\alpha > 0$.*
3. *The law of $t^{-1} Z_1 | x_2 \geq -t$ converges as $t \to 0+$.*
4. $\sup_{n \geq 1} n \log \mathbf{E}(\int e^{a_n x} Z_1(\mathrm{d}x)) < \infty$, *with a_n such that $\mathbf{P}(x_2 \geq -a_n) = 1/n$.*

Then $(a_n^{-1} Z_{\lfloor nt \rfloor}, t \geq 0)$ converges in law to an α-stable branching Lévy process.

Plan

1. Infinitely ramified point measures and branching Lévy processes

2. Stable branching Lévy processes

3. Fixed points of the branching convolution equation

Fixed point of the branching convolution equation

Branching-invariant point measures

Definition

Let Z be a random point measure. A point measure E is said to be branching-invariant with respect to the measure Z if

$$E \overset{(d)}{=} \sum_{j \in \mathbb{N}} \tau_{z_j} E_j,$$

where $(E_j, j \geq 1)$ are i.i.d. copies of E, and $(z_j, j \geq 1)$ are the atoms of Z.

Branching convolution equation

Writing $\mathcal{E}$ the law of E and $\mathcal{Z}$ the law of Z, we have

$$\mathcal{E} = \mathcal{Z} \circledast \mathcal{E}.$$

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Assumptions on the law $\mathcal{Z}$

A1 (supercriticality)

$\mathbf{E}(Z(\mathbb{R})) > 1$ and there exists $\theta > 0$ such that $\mathbf{E}\left(\sum_{j\in\mathbb{N}} e^{\theta z_j}\right) = 1$.

A2 (non-lattice)

For all $a, b > 0$, we have $\mathbf{P}(\mathrm{Supp}(Z) \subset a + b\mathbb{Z}) < 1$.

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Fixed point of the branching convolution equation

Regular or boundary case

Write $X = \sum_{j\in\mathbb{N}} e^{\theta z_j}$ and $\tilde{X} = \sum_{j\in\mathbb{N}} (z_j)_+ e^{\theta z_j}$.

A3a (regular case)

$\mathbf{E}\left(\sum_{j\in\mathbb{N}} z_j e^{\theta z_j}\right) \in (-\infty, 0)$ and $\mathbf{E}(X \log_+(X)) < \infty$.
In this case we write $S = \lim_{n\to\infty} \int e^{\theta x} Z_n(\mathrm{d}x)$ a.s.

A3b (boundary case)

$\mathbf{E}\left(\sum_{j\in\mathbb{N}} z_j e^{\theta z_j}\right) = 0$, $\mathbf{E}\left(\sum_{j\in\mathbb{N}} z_j^2 e^{\theta z_j}\right) < \infty$ and
$\mathbf{E}(X \log_+(X)^2) + \mathbf{E}(\tilde{X} \log_+(\tilde{X})) < \infty$.
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We refer to A3 as A3a or A3b.

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In this case we write $S = -\lim_{n\to\infty} \int x e^{\theta x} Z_n(\mathrm{d}x)$ a.s.

We refer to A3 as A3a or A3b.

Fixed point of the branching convolution equation

Regular or boundary case

Write $X = \sum_{j\in\mathbb{N}} e^{\theta z_j}$ and $\tilde{X} = \sum_{j\in\mathbb{N}} (z_j)_+ e^{\theta z_j}$.

A3a (regular case)

$\mathbf{E}\left(\sum_{j\in\mathbb{N}} z_j e^{\theta z_j}\right) \in (-\infty, 0)$ and $\mathbf{E}(X \log_+(X)) < \infty$.
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A3b (boundary case)

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Factorization of the fixed point

Theorem (Maillard, M. (2021+))

Under assumptions A1, A2 and A3, any law $\mathcal{E}$ satisfying $\mathcal{E} = \mathcal{Z} \circledast \mathcal{E}$ can be written as

$$\mathcal{E} = \mathcal{X}_c \circledast \mathcal{D},$$

where $\mathcal{X}_c$ is the law of a Poisson point process with intensity $cSe^{-\theta x}\mathrm{d}x$, and $\mathcal{D}$ is the law of a point measure with support in

$$\mathfrak{P}^* = \{ \mathbf{x} \in \mathfrak{P} : x_1 = 0, x_2 \leq 0\}.$$

In other words, a point measure satisfying $E \overset{(d)}{=} \sum_{j \in \mathbb{N}} \tau_{z_j} E_j$ is a decorated Poisson point process with exponential intensity, shifted by $\theta^{-1} \log(cS)$.

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Fixed point of the branching convolution equation

A possible application

Remark

The branching convolution equation does not allows the characterization of the law of the decoration. Indeed, if $\mathcal{E}$ satisfies the branching convolution equation, then so does $\mathcal{E} \circledast \mathcal{X}$ for any law $\mathcal{X}$ on $\mathfrak{P}^*$.

Writing $(Z_n, n \geq 0)$ a branching random walk with reproduction law $\mathcal{Z}$, Madaule (2016) proved that $\tau_{-m_n} Z_n$ converges in law, with $m_n \sim_{n\to\infty} nv$.
Using the branching property of the branching random walk, we observe that $\mathcal{E} = \mathcal{Z} \circledast \mathcal{E}$, therefore the limit has to be a decorated Poisson point process with exponential integrability.
This result could be used in time-inhomogeneous branching random walks to obtain the law of the limit in a similar fashion.

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Fixed point of the branching convolution equation

Steps of the proof

Consider a random point measure satisfying $E \overset{(d)}{=} \sum_{j\in\mathbb{N}} \tau_{z_j} E_j$.

- For all test functions ϕ, we set

$$F_\phi(z) = \mathbf{E}\left(\exp\left(-\sum_{j\in\mathbb{N}} \phi(z_j + z)\right)\right) = \mathbf{E}\left(e^{-\int \phi(\cdot+z)\mathrm{d}E}\right).$$

- By the branching convolution equation $F_\phi(z) = \mathbf{E}\left(\prod_{j\in\mathbb{Z}} F_\phi(z+z_j)\right)$ hence $F_\phi \circ \log$ is a fixed point of the smoothing transform.
- Using Alsmeyer, Biggins, Meiners (2012), there exists $c(\phi)$ such that

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- By Subag, Zeitouni (2015), if $\mathbf{E}\left(e^{-\int \phi(\cdot+z)\mathrm{d}E}\right) = \mathbf{E}\left(e^{-c(\phi)Se^{-\theta z}}\right)$ for enough test functions, then E is a decorated Poisson point process with exponential intensity, shifted by $\theta^{-1}\log(cS)$.

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The reciprocal branching convolution equation

Theorem (Chen, Garban, Shekhal (2020+))

Let $\mathcal{Z}_t$ be the law of the value at time t of a branching Brownian motion with drift $b < -\sqrt{2}$. A point measure distribution $\mathcal{E}$ such that for all $t \geq 0$, $\mathcal{E} = \mathcal{E} \circledast \mathcal{Z}$ can be written

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Remark

Therefore, the direct branching convolution equation characterizes the random shift, and the reciprocal branching convolution equation should characterize the point measure.

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Conjectures

The reciprocal branching convolution equation

A point measure distribution $\mathcal{E}$ satisfying $\mathcal{E} = \mathcal{E} \circledast \mathcal{Y}$ is the law of a decorated Poisson point process with exponential intensity, with a fixed decoration $\mathcal{D}(\mathcal{Y})$, but a shift that can be any random variable S.

Two-sided branching convolution equation

A point measure distribution $\mathcal{E}$ satisfying $\mathcal{E} = \mathcal{Z} \circledast \mathcal{E} = \mathcal{E} \circledast \mathcal{Y}$ is the law of a decorated Poisson point process with exponential intensity, with decoration $\mathcal{D}(\mathcal{Y})$ and shift $cS(\mathcal{Z})$, for some $c > 0$.

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Thank you for your attention!