Limit theorems for clocks

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Joint work with
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July 8, 2021

Random excursion with Jean Bertoin, Paris

Outline

pssMp

A positive self-similar Markov process (pssMp) with self-similarity index $\alpha > 0$ is a strong Markov process $(X, \mathbb{Q}_a, a > 0)$ with paths the Skorokhod space $\mathbb{D}(\mathbb{R}^+; \mathbb{R}^+)$ which satisfies the scaling property :

$$(\{cX_{c^{-\alpha}t}, t \geqslant 0\}, \mathbb{Q}_a) \overset{(d)}{=} (\{X_t, t \geqslant 0\}, \mathbb{Q}_{ca}) \tag{1}$$

for every $a, c > 0$.

Examples : $\alpha = 1$

- $\mathrm{BESQ}(d)$: $B_1^2 + \cdots + B_d^2$
- modulus of a multi-dimensional Cauchy process.

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Lamperti (1972)

Every pssMp X never reaching 0 can be expressed as the exponential of a Lévy process not drifting to $-\infty$, time-changed by the inverse of its exponential functional. Formally, if $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α never reaching 0, set

$$H^{(X)}(t) = \int_0^t \frac{ds}{X_s^\alpha}\,,\ (t \geqslant 0) \tag{2}$$

and define $A^{(X)}$ its inverse by

$$A^{(X)}(t) = \inf\{u \geqslant 0 : H^{(X)}(u) \geqslant t\}\,,\ (t \geqslant 0)\,,$$

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Let ξ be the process defined by

$$\xi_t = \log X_{A^{(X)}(t)} - \log X_0 \ , \ (t \geqslant 0) \, .$$

Then, for every $a > 0$, the distribution of $(\xi_t, t \geqslant 0)$ under $\mathbb{Q}_a$ does not depend on a and is the distribution of a Lévy process starting from 0. The quantity

$$H^{(X)}(t) = \int_0^t \frac{ds}{X_s^\alpha} \ , \ (t \geqslant 0)$$

is called the clock of X. In short

$$X_t = X_0 \exp \xi_{H^{(X)}(t)} \, .$$

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Conversely, let $(\xi_t, t \geqslant 0)$ be a Lévy process starting from 0 and not drifting to $-\infty$, and let $\mathbb{P}$ be the underlying probability.
Fix $\alpha > 0$. Set

$$\mathcal{A}^{(\xi)}(t) = \int_0^t e^{\alpha \xi_s} ds . \tag{3}$$

and $\tau^{(\xi)}$ the inverse process

$$\tau^{(\xi)}(t) = \inf\{u \geqslant 0 : \mathcal{A}^{(\xi)}(u) \geqslant t\} , \ (t \geqslant 0) .$$

For every $a > 0$, let $\mathbb{Q}_a$ be the distribution under $\mathbb{P}$ of

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then $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α not reaching 0 and we have

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Entrance boundary

If ξ is non-lattice and if $0 < \mathbb{E}\xi_1 < \infty$, we can define $\mathbb{Q}_0$ under which

$$(bX_{b^{-\alpha}t}, t \geqslant 0) \stackrel{(d)}{=} (X_t, t \geqslant 0)$$

Actually

$$\mathbb{Q}_0 = \lim_{a \downarrow 0} \mathbb{Q}_a$$

We have

$$\mathbb{E}_0 f(X_1) = \frac{1}{\alpha \mathbb{E}\xi_1} \mathbb{E}\left[I_\infty^{-1} f\left(I_\infty^{-1/\alpha}\right)\right] . \tag{5}$$

$$\text{with } \; I_\infty = \int_0^\infty e^{-\alpha \xi_s} ds \, , \tag{6}$$

(perpetuity).

Ornstein-Uhlenbeck

To a pssMp (X_t) of index α, we associate classically a generalized Ornstein-Uhlenbeck (OU) process

$$U(t) = e^{-t/\alpha} X(e^t) \tag{7}$$

The generators of X and U are connected by

$$L^U h(x) = L^X h(x) - \frac{x}{\alpha} h'(x) \,. \tag{8}$$

Under $\mathbb{Q}_0$, the process U is strictly stationary, Markovian, ergodic , and its invariant measure is

$$\mu^U : f \mapsto \mathbb{E}_0 f(X_1) \,.$$

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Let ψ be the Laplace exponent of (ξ_t). It is a convex function $(-\infty, \infty]$-valued given by

$$\mathbb{E}(\exp m\xi_t) = \exp(t\psi(m)) \quad (m \in \mathbb{R})\,. \tag{9}$$

and dom $\psi := \{m : \psi(m) < \infty\}$ is an interval which contains 0. We assume

$$\text{int dom}\ \psi =: (m_-, m_+) \ni 0 \tag{10}$$

$$\mathbb{E}\xi_1 =: \psi'(0) > 0\,. \tag{11}$$

(ψ is analytical on (m_-, m_+)). We set

$$m_0 = \inf\{\theta : \psi'(\theta) > 0\}\ , \tau_+ = \frac{1}{\psi'(m_+)}\ ,\ \tau_0 = \frac{1}{\psi'(m_0)}\ ,$$

where $1/\psi'(\pm\infty) := \lim_{m\to\pm\infty} m/\psi(m)$. We assume also

$$\xi \text{ is non-lattice}\,. \tag{12}$$

Outline

A) Bessel clock

$$\xi_t = 2B_t + 2\nu t\ ,\ X_t = \mathrm{BESQ}(2(1+\nu))$$

$$\psi(m) = 2m(m+\nu)\ ,\ I_\infty^{-1} \stackrel{(d)}{=} 2\mathrm{Gamma}(\nu)\,.$$

B) Poissonian Examples

1) $\xi_t = dt + \mathrm{Pois}(a, b)_t$

$$\psi(m) = m\left(d + \frac{a}{b-m}\right)\ ,\ I_\infty \stackrel{(d)}{=} a^{-1}\mathrm{Beta}(1+b, a\alpha^{-1})$$

2) $\xi_t = -t + \mathrm{Pois}(a, b)_t$ with $b < a$

$$\psi(m) = m\left(-1 + \frac{a}{b-m}\right)\ ,\ I_\infty \stackrel{(d)}{=} \mathrm{Beta}_2(1+b, a-b)$$

3) $\xi_t = t - \mathrm{Pois}(a, b)_t$ (“saw-tooth process”) with $a < b$

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C) Process $X^{\uparrow}$ spectrally negative conditioned to stay positive.
Let $\alpha \in (1,2)$.

$$\psi(m) = \frac{\Gamma(m+\alpha)}{\Gamma(m)}\ ,\ I_\infty \overset{(d)}{=} S_{1/\alpha}(1)$$

D) Hypergeometric stable process
X : modulus of a d-dimensional symmetric α-stable process $(d > \alpha)$.

$$\psi(m) = -2^{\alpha}\frac{\Gamma((-m+\alpha)/2)}{\Gamma(-m/2)}\frac{\Gamma((m+d)/2)}{\Gamma((m+d-\alpha)/2)}\ ,\ (m \in (-d,\alpha))\,.$$

Particular case : $\alpha = 1$ $d = 3$,

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E) CBI process index κ

branching mechanism $\varphi(u) = -u^{\kappa+1}$
immigration mechanism $\chi(u) = \delta(\kappa+1)u^{\kappa}$

$$\psi(m) = (\kappa - (\kappa+1)\delta - m)\frac{\Gamma(-m+\kappa)}{\Gamma(-m)}\ ,\ (m \in (-\infty, \kappa))$$

Outline

Law of Large Numbers

Theorem

Assume (10) and (12). As $t \to \infty$*,*

1) *for all* $a > 0$*,*

$$\frac{1}{\log t} \int_0^t \frac{ds}{X_s^\alpha} \to (\alpha p)^{-1}, \quad \mathbb{Q}_a - \textit{a.s.}$$

2)

$$\frac{1}{\log t} \int_1^t \frac{ds}{X_s^\alpha} \to (\alpha p)^{-1}, \quad \mathbb{Q}_0 - \textit{a.s.}$$

Sketch of proof of 2) : By the ergodic theorem, we have that for $f \in L^1(\mu)$:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(U_s)ds = \mu^U(f)\,, \quad \mathbb{Q}_0 - \text{a.s.}$$

But (definition + change $e^s = r$))

$$\int_0^T f(U_s)ds = \int_0^T f(e^{-s/\alpha}X(e^s))ds = \int_1^{e^T} f(r^{-1/\alpha}X(r))r^{-1}dr \qquad (13)$$

so that taking $T = \log t$,

$$\lim_{t\to\infty} \frac{1}{\log t}\int_1^t f(r^{-1}X(r))r^{-1}dr = \mu^U(f)\,, \quad \mathbb{Q}_0 - \text{a.s.}$$

For $f(x) = x^{-\alpha}$ we get

$$\lim_{t\to\infty} \frac{1}{\log t}\int_1^t \frac{du}{X_u^\alpha} = (\alpha p)^{-1}\,, \quad \mathbb{Q}_0 - \text{a.s.}$$

Outline

Theorem (Caballero-R '20)

1) *Under* $\mathbb{Q}_0$*, as* $T \to \infty$*,*

$$\left((\log T)^{-1/2} \left(\int_1^{T^t} \frac{dr}{X^\alpha(r)} - \frac{t \log T}{\alpha p} \right); t \geqslant 0 \right) \Rightarrow (\nu W(t); t \geqslant 0) \tag{14}$$

$$p = \psi'(0)\ ,\ \nu^2 = \frac{\sigma^2}{\alpha p^3},\ \sigma^2 = \psi''(0)\,,$$

where $(W_t, t \geqslant 0)$ *is a standard Brownian motion.*

2) *If one of the following conditions is satisfied*
   - 2.1 $\exists m > 0 : \psi(m) < 0$,
   - 2.2 $\exists m > \alpha : \psi(m) > 0$,

   then for all $a > 0$*, under* $\mathbb{Q}_a$*, as* $T \to \infty$*, (14) holds true.*

Sketch of proof of (1)

Theorem (Bhattacharya 1982 Th. 2.1)

Let (Y_t) *be a stationary ergodic process with invariant probability* π*. If* f *is in the range of the extended generator* $\hat{A}$ *of* (Y_t)*, then, as* $n \to \infty$

$$\left(n^{-1/2}\int_0^{nt} f(Y_s)ds\right)_{t\geqslant 0} \Rightarrow (\rho W(t))_{t\geqslant 0} \tag{15}$$

where

$$\rho^2 = -2\int f(x)g(x)\pi(dx)\ ,\ \hat{A}g = f\,. \tag{16}$$

Remark about generators.

$$L^X h(x) = x^{-\alpha} L^{\xi}(h \circ \exp)(\log x)$$

and if $f_m(x) = x^m$ then

$$L^X f_m = \psi(m) f_{m-1}\,.$$

For (1) (“quenched FCLT”), the proof is more involved. We have to consider exponential ergodicity and some appropriate criterion.
In particular, for the CBI process, we have to consider another OU process (already considered by Bertoin)

$$\tilde{U}(t) = e^{-t/\kappa} X(e^t - 1)$$

which is a CBI with imm. mechanism χ and with br. mechanism $\tilde{\varphi}(\lambda) = \varphi(\lambda) + \kappa^{-1}\lambda$.

Outline

Large Deviations

Set

$$m_0 = \inf\{\theta : \psi'(\theta) > 0\}\ , \tau_+ = \frac{1}{\psi'(m_+)}\ ,\ \tau_0 = \frac{1}{\psi'(m_0)} \text{ and } \Delta = (\tau_+, \tau_0)\,,$$

where $1/\psi'(\pm\infty) := \lim_{m\to\pm\infty} m/\psi(m)$.

Theorem (Demni-R-Zani '15)

Let $a > 0$. *For every* $x \in \bar{\Delta}$ *we have*

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{Q}_a\left(\frac{H^{(X)}(t)}{\log t} \in [x-\varepsilon, x+\varepsilon]\right) = -\mathcal{J}(x) \tag{17}$$

with

$$\mathcal{J}(x) = \sup_{m\in(m_0,m_+)} \{m - x\psi(m)\} = x\,\psi^*(1/x)\,. \tag{18}$$

The rate function $\mathcal{J}$ is positive, convex and reach its unique minimum 0 at $1/\psi'(0) = 1/p = 1/\mathbb{E}\xi_1$.

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Sketch of proof : Gartner-Ellis.

Lemma

For $\theta \in (-\psi(m_+), -\psi(m_0))$*, as* $t \to \infty$ *we have*

$$\frac{1}{\log t} \log \mathbb{E}_a \exp(\theta H^{(X)}(t)) \to L(\theta) \tag{19}$$

with

$$L(\theta) = -m \Longleftrightarrow \theta = -\psi(m) . \tag{20}$$

Sketch of proof of the lemma : define an auxiliary process via Girsanov. For $m \in (m_-, m_+)$ let

$$\psi_m(\theta) = \psi(m+\theta) - \psi(m) ,$$

and let $\{(\xi_t, t \geqslant 0); \mathbb{P}^m\}$ be a Lévy process starting from 0 with exponent ψ_m (Esscher transform). At last, let $\{(X_t, t \geqslant 0); (\mathbb{Q}_a^m)_{a>0}\}$ be the associated pssMp .

From the remark about generators, we have

$$\mathbb{Q}_a^m|_{\mathcal{F}_t} = \left(\frac{X_t}{a}\right)^m \exp\left(-\psi(m)\int_0^t \frac{ds}{X_s}\right) . \mathbb{Q}_a|_{\mathcal{F}_t} \quad , \ (t \geqslant 0) . \tag{21}$$

hence

$$\mathbb{E}_a \exp\left(-\psi(m)\int_0^t \frac{ds}{X_s}\right) = a^m \mathbb{E}_a^m[(X_t)^{-m}] \tag{22}$$

and, due to the scaling property

$$\mathbb{E}_a \exp\left(-\psi(m)\int_0^t \frac{ds}{X_s}\right) = a^m t^{-m} \mathbb{E}_{a/t}^m[(X_1)^{-m}] . \tag{23}$$

hence, as $t \to \infty$,

$$\log \mathbb{E}_a \exp\left(-\psi(m)\int_0^t \frac{ds}{X_s}\right) = -m \log t + m \log a + \log \mathbb{E}_0^m[(X_1)^{-m}] + o(1)$$

Explicit rate functions :

A) (Bessel clock) : $I(x) = \frac{(1-2\nu x)^2}{8x}$, $(x > 0)$.

Poissonian examples

B) $I(x) = \left(\sqrt{b(1-dx)} - \sqrt{ax}\right)^2$, $(0 \leqslant x \leqslant d^{-1})$

C) $I(x) = \left(\sqrt{b(1+x)} - \sqrt{ax}\right)^2$, $(x \geqslant 0)$

D) $I(x) = \left(\sqrt{b(x-1)} - \sqrt{ax}\right)^2$, $(x \geqslant 1)$

Comments

1) The use of f_m seems tricky
2) We could think of applying an LDP for OU and then pushforwarding it to the clock. For stationary processes, there are LDPs at level 2 (occupation measure) and at level 3 (process).

Indeed,

$$\frac{1}{\log T}\int_1^T \frac{dr}{X_r^\alpha} = \frac{1}{\log T}\int_0^{\log T} \frac{ds}{U_s^\alpha} = \left\langle \frac{1}{\log T}\int_0^{\log T} \delta_{U_s} ds \ , f_{-\alpha} \right\rangle \tag{24}$$

The issue is that $f_{-\alpha}$ is <u>not continuous</u>, so we cannot apply a contraction.

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Let us look at the simplest case (Bessel clock). Assume that $2(1+\nu) = d$ is an integer. Then the generator of U is the Laguerre one :

$$L^{U}(h) = 2xh'' + (d-x)h',$$

μ^{U} is the χ^2_d and the rate function for the occupation measure is

$$I_2(\mu) = -\int_0^\infty h(x)L^{U}(h)(x)\mu^{U}(dx)$$

with $h^2 = d\mu/d\mu^{U}$.
The variational problem

$$\inf\{-\int h(x)L^{U}(h)(x)\mu^{U}(dx); \int h^2(x)f_{-1}(x)\mu^{U}(dx) = y, \int h^2(x)\mu^{U}(dx) = 1$$

via Lagrange multipliers has a unique admissible solution

$$h(x) = Cx^m$$

where $\psi'(m) = y^{-1}$. It tells us that the optimal occupation measure is $h^2\mu^{U}$. Moreover the distribution $\mathbb{Q}_a^m$ could be interpreted as optimal for the third level LDP.

Thanks for your attention !

Happy birthday Jean !