Excursions away from the Lipschitz minorant of a Lévy process

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Outline

1. Lipschitz minorants and the contact set
2. The process after the first positive point in the contact set
3. Space-time regenerative systems
4. Brownian motion with drift

α-Lipschitz minorants

- A function $f : \mathbb{R} \to \mathbb{R}$ is α-Lipschitz if $|f(s) - f(t)| \leq \alpha|s - t|$.
- If a family $\{f_\theta\}_{\theta \in \Theta}$ of α-Lipschitz functions is uniformly bounded above on compacts, then $t \mapsto \sup_{\theta \in \Theta} f_\theta(t)$ is α-Lipschitz.
- So, if f is bounded below on compacts,

$$\liminf_{t \to -\infty} f(t) - \alpha t > -\infty \quad \text{and} \quad \liminf_{t \to +\infty} f(t) + \alpha t > -\infty,$$

  then there is a unique greatest $m : \mathbb{R} \to \mathbb{R}$ such that $m \leq f$ and m is α-Lipschitz.
- Call m the α-Lipschitz minorant of f.
- Define the contact set : $\mathcal{Z}_f := \{t \in \mathbb{R} : f(t) \wedge f(t-) = m(t)\}$ when f is càdlàg.

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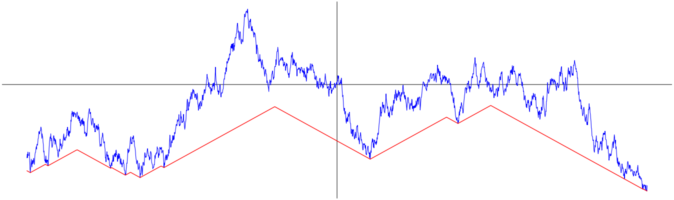
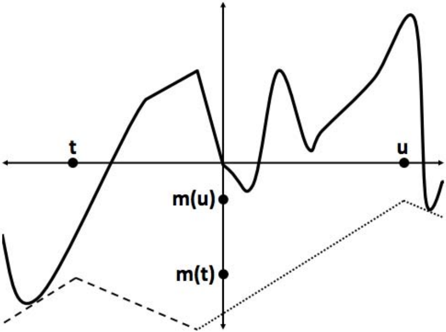
What does the minorant look like?



Finding the minorant

It is easy to see that

$$\begin{aligned} m(t) &= \sup\{h \in \mathbb{R} : h - \alpha|t-s| \le f(s) \text{ for all } s \in \mathbb{R}\} \\ &= \inf\{f(s) + \alpha|t-s| : s \in \mathbb{R}\}. \end{aligned}$$



The minorant away from the contact set ("sawteeth")

Lemma

Suppose that $f : \mathbb{R} \to \mathbb{R}$ *is a càdlàg with* α*-Lipschitz minorant* $m : \mathbb{R} \to \mathbb{R}$. *The set* $\{t \in \mathbb{R} : m(t) = f(t) \wedge f(t-)\}$ *is closed. If* $t' < t''$ *are such that* $f(t') \wedge f(t'-) = m(t')$, $f(t'') \wedge f(t''-) = m(t'')$, *and* $f(t) \wedge f(t-) > m(t)$ *for all* $t' < t < t''$, *then setting* $t^* = (f(t'') \wedge f(t''-) - f(t') \wedge f(t'-) + \alpha(t'' + t'))/(2\alpha)$, *we have*

$$m(t) = \begin{cases} f(t') \wedge f(t'-) + \alpha(t - t'), & t' \le t \le t^*, \\ f(t'') \wedge f(t'') + \alpha(t'' - t), & t^* \le t \le t''. \end{cases}$$

Construction of the first positive contact point

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ *be a càdlàg function with* α*-Lipschitz minorant* $m : \mathbb{R} \to \mathbb{R}$*. Set*

$$\mathbf{d} := \inf\{t > 0 : f(t) \wedge f(t-) = m(t)\},$$

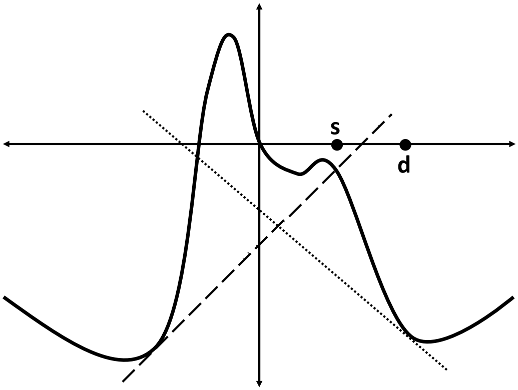
$$\mathbf{s} := \inf\left\{t > 0 : f(t) \wedge f(t-) - \alpha t \leq \inf\{f(u) - \alpha u : u \leq 0\}\right\},$$

and

$$\mathbf{e} := \inf\{t \geq \mathbf{s} : f(t) \wedge f(t-) + \alpha(t - \mathbf{s}) = \inf\{f(u) + \alpha(u - \mathbf{s}) : u \geq \mathbf{s}\}\}.$$

Suppose that $f(\mathbf{s}) \leq f(\mathbf{s}-)$*. Then,* $\mathbf{e} = \mathbf{d}$*.*

Construction of the first positive contact point



α-Lipschitz minorants of a Lévy processes

- We focus on the α-Lipschitz minorant of a two-sided Lévy process $X = (X_t)_{t\in\mathbb{R}}$ with $X_0 = 0$ (that is, X has càdlàg paths and, for any $t \in \mathbb{R}$, $(X_{u+t} - X_t)_{u\geq 0}$ is independent of $(X_s, -\infty < s \leq t)$ with a distribution that doesn't depend on t).
- The α-Lipschitz minorant exists $\iff \mathbb{E}[|X_1|] < \infty$ and $|\mathbb{E}[X_1]| < \alpha$ (excluding the case of a trivial drift $\pm\alpha$ process); for example, Brownian motion with drift β, where $|\beta| < \alpha$.
- Let $(M_t)_{t\in\mathbb{R}}$ be the α-Lipschitz minorant of the Lévy process $X = (X_t)_{t\in\mathbb{R}}$.

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Contact set for a Lévy process

- Define the contact set

$$\mathcal{Z} := \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}.$$

- The random closed set $\mathcal{Z}$ is a stationary and regenerative (we will see why soon).
- Hence, $\mathcal{Z}$ is the closed range of a subordinator 'in stationarity' (unique up to a speed factor).

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Characteristic exponent of the Lévy process X

The distribution of X is characterized by the Lévy-Khintchine formula $\mathbb{E}[e^{i\theta(X_t - X_s)}] = e^{-(t-s)\Psi(\theta)}$ for $\theta \in \mathbb{R}$ and $-\infty < s \leq t < \infty$, where:

$$\Psi(\theta) = -ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x| \leq 1\}})\, \Pi(dx)$$

with $a \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and Π a σ-finite measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\, \Pi(dx) < \infty$.

Bounded variation case

The sample paths of X have bounded variation almost surely if and only if $\sigma = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|) \, \Pi(dx) < \infty$. In this case Ψ can be rewritten as

$$\Psi(\theta) = -id\theta + \int_{\mathbb{R}} (1 - e^{i\theta x}) \, \Pi(dx).$$

We call $d \in \mathbb{R}$ the drift coefficient.

The Lebesgue measure of $\mathcal{Z}$

The random closed set $\mathcal{Z}$ either has infinite Lebesgue measure almost surely or zero Lebesgue measure almost surely.

- If the sample paths of X have unbounded variation almost surely, then $\mathcal{Z}$ has zero Lebesgue measure almost surely.
- If X has sample paths of bounded variation and $|d| > \alpha$, then $\mathcal{Z}$ has zero Lebesgue measure almost surely.
- If X has sample paths of bounded variation and $|d| < \alpha$, then $\mathcal{Z}$ has infinite Lebesgue measure almost surely.
- If X has sample paths of bounded variation and $|d| = \alpha$, then whether the Lebesgue measure of $\mathcal{Z}$ is infinite or zero is determined by an integral condition involving the Lévy measure Π that we omit. In particular, if $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$, and $|d| = \alpha$, then the Lebesgue measure of $\mathcal{Z}$ is almost surely infinite.

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When is $\mathcal{Z}$ discrete?

If $\mathcal{Z}$ has zero Lebesgue measure, then $\mathcal{Z}$ is either almost surely a discrete set or almost surely a perfect set with empty interior.

- If $\sigma > 0$, then $\mathcal{Z}$ is almost surely discrete.
- If $\sigma = 0$ and $\Pi(\mathbb{R}) = \infty$, then $\mathcal{Z}$ is almost surely discrete if and only if

$$\int_0^1 t^{-1}\mathbb{P}\{X_t \in [-\alpha t, \alpha t]\}\, dt < \infty.$$

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Notation

For $t \in \mathbb{R}$ set $G_t := \sup(\mathcal{Z} \cap (-\infty, t))$ and $D_t := \inf(\mathcal{Z} \cap (t, +\infty))$. Put $G := G_0$ and $D := D_0$.

Remark

We have from the construction of the first positive contact point that

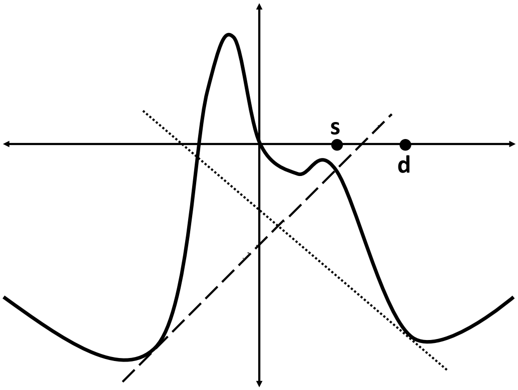
$$D = \inf\{t \geq S : X_t \wedge X_{t-} + \alpha t = \inf\{X_u + \alpha u : u \geq S\}\},$$

where

$$S = S_0 := \inf\{s > 0 : X_s \wedge X_{s-} - \alpha s \leq \inf\{X_u - \alpha u : u \leq 0\}\}$$

because almost surely $X_S \leq X_{S-}$.

Reminder



Some terminology

Say for a Lévy process $(Y_t)_{t\in\mathbb{R}}$ that 0 is regular for $(0,\infty)$ if almost surely Y visits $(0,\infty)$ at arbitrarily small positive times; for example, 0 is regular for $(0,\infty)$ for Brownian motion with arbitrary drift.

Theorem

Suppose for the Lévy process $(X_t + \alpha t)_{t \in \mathbb{R}}$ that 0 is regular for $(0, \infty)$. The process $(X_{t+D} - X_D)_{t \geq 0}$ is independent of $(X_t, -\infty < t \leq D)$. Moreover, the process $(X_{t+D} - X_D + \alpha t)_{t \geq 0}$ is Markovian with the same distribution as the process $(X_t + \alpha t)_{t \geq 0}$ conditioned to stay positive in the sense of a Doob h-transform.

Note: This is not just the strong Markov property for X; the time D is not a stopping time for X. The proof uses a path decomposition of Millar for a Markov process at the time of its global minimum that follows from general results by Pittenger & Shih and Getoor & Sharpe about path decompositions at co-terminal times.

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Space-time regenerative property

Recall that $D_t := \inf(\mathcal{Z} \cap (t, +\infty))$.

Theorem

Suppose for the Lévy process $(X_t + \alpha t)_{t \in \mathbb{R}}$ that 0 is regular for $(0, \infty)$. For all $t \in \mathbb{R}$, the pair $((X_{u+D_t} - X_{D_t})_{u \geq 0}, \mathcal{Z} \cap [D_t, +\infty) - D_t)$ is independent of the past of $(X, \mathcal{Z})$ up to time D_t and has a distribution that does not depend on t.

We will see why in the next few slides.

Exploiting "stationarity"

Lemma

Consider càdlàg $f : \mathbb{R} \to \mathbb{R}$. Put $g := a + f(b + \cdot)$. Write m_f and m_g for the α-Lipschitz minorants of f and g. Write $\mathcal{Z}_f$ and $\mathcal{Z}_g$ for the corresponding contact sets. Then $m_g = a + m_f(b + \cdot)$ and $\mathcal{Z}_g = \mathcal{Z}_f - b$.

- We have $(X_s)_{s\in\mathbb{R}} \stackrel{d}{=} (X_{s+t} - X_t)_{s\in\mathbb{R}}$ for any $t \in \mathbb{R}$.
- So, to show the theorem, it suffices to show that pair $((X_{u+D} - X_D)_{u\geq 0}, \mathcal{Z} \cap [D, +\infty) - D)$ is independent of the past of $(X, \mathcal{Z})$ up to time D (recall $D := D_0$).

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Exploiting "locality"

Lemma

Consider a function $f : \mathbb{R} \to \mathbb{R}$ for which the α-Lipschitz minorant m exists. Fix $a \in \mathbb{R}$ such that $m(a) = f(a)$. Define $f^{\rightarrow} : \mathbb{R} \to \mathbb{R}$ by

$$f^{\rightarrow}(t) = \begin{cases} f(a) + \alpha(t-a), & t \le a, \\ f(t), & t > a. \end{cases}$$

Denote the α-Lipschitz minorant of $f^{\rightarrow}$ by $m^{\rightarrow}$. Then $m(t) = m^{\rightarrow}(t)$ for all $t \ge a$.

- So, to prove the theorem, it suffices to recall that $(X_{u+D} - X_D)_{u \ge 0}$ is independent of $(X_s, -\infty < s \le D)$.

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Consider a function $f : \mathbb{R} \to \mathbb{R}$ *for which the* α*-Lipschitz minorant* m *exists. Fix* $a \in \mathbb{R}$ *such that* $m(a) = f(a)$*. Define* $f^{\to} : \mathbb{R} \to \mathbb{R}$ *by*

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Stopping time extension

Corollary

Suppose for the Lévy process $(X_t + \alpha t)_{t\in\mathbb{R}}$ that 0 is regular for $(0,\infty)$. If T is an $\mathbb{R}$-valued stopping time for a suitable filtration, then the pair $((X_{u+D_T} - X_{D_T})_{u\geq 0}, \mathcal{Z} \cap [D_T, +\infty) - D_T)$ is independent of the past of $(X, \mathcal{Z})$ up to time D_T and has a distribution that does not depend on T.

When $\mathcal{Z}$ is discrete

Let $0 < T_1 < T_2 < \ldots$ be the successive points in $\mathcal{Z} \cap (0, +\infty)$. Put

$$Y_t^n := X_{(t+T_n) \wedge T_{n+1}} - X_{T_n}, \quad t \geq 0.$$

Then the sequence of paths $(Y^n)_{n \in \mathbb{N}}$ is independent and identically distributed.

When $\mathcal{Z}$ is not discrete

When $\mathcal{Z}$ is not discrete there is a local time on $\mathcal{Z}$ and we can construct a Poisson random measure on the set $\mathbb{R} \times \{\text{stopped paths that start at } 0\}$ that records the excursions away from the contact set and the order in which they occur.

Outline

Theorem (Williams 74')

Let $\mu > 0$. On some probability space, take three independent random elements:

- $(B_t^{(-\mu)},\ t \geq 0)$ *a BM with drift* $-\mu$*;*
- $(R_t^{(\mu)}, t \geq 0)$ *a diffusion that is solution of the following SDE*

$$dR_t^{(\mu)} = dB_t + \mu \coth(\mu R_t^{(\mu)})dt, \quad R_0^{(\mu)} = 0,$$

  where B *is a standard Brownian motion;*
- γ *an exponential r.v with rate* 2μ*.*

Set $\tau = \inf\{t \geq 0 : B_t^{(-\mu)} = -\gamma\}$ *and*

$$H_t = \begin{cases} B_t^{(-\mu)}, & 0 \leq t \leq \tau, \\ R_{t-\tau}^{(\mu)} - \gamma, & t \geq \tau. \end{cases}$$

Then, $(H_t)_{t\geq 0}$ *is a Brownian motion with drift* μ*.*

3-dim Bessel process with drift

The diffusion $R^{(\mu)}$ is called a 3-dimensional Bessel process with drift μ, denoted $\mathrm{BES}(3, \mu)$ (it is the radial part of a 3-dimensional Brownian motion with drift vector having magnitude μ).

Connection with minorants

From now on, take $(X_t)_{t\in\mathbb{R}}$ to be a Brownian motion with drift β for $|\beta| < \alpha$.

By the Williams path decomposition,

$$(X_{t+D} - X_D,\, t \geq 0) = (R_t^{(\alpha+\beta)} - \alpha t,\, t \geq 0),$$

where $R^{(\alpha+\beta)} \stackrel{d}{=} \mathrm{BES}(3, \alpha + \beta)$.

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Features of a generic excursion

Write $(X_{t+D} - X_D,\ 0 \le t \le \zeta)$ for the first positive time excursion away from the minorant (here $D = T_1$ and $\zeta = T_2 - T_1$ in previous notation).

Recall that the distribution of this excursion is the same as that of any other excursion except the one straddling time 0.

We are interested in the distribution of

- the lifetime ζ;
- the time $L = \operatorname{argmax}\{M_{t+D} : 0 \le t \le \zeta\}$ at which the minorant has its peak during the excursion.
- the final value $X_{\zeta+D} - X_D = M_{\zeta+D} - M_D$ of the excursion.

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Theorem

- *A pair of random variables* $(\tau, \hat{\gamma})$ *with the joint density*

$$f_{\tau,\hat{\gamma}}(t,x) = \frac{\exp\left(-\frac{(\alpha-\beta)^2 t}{2} - 2(\alpha+\beta)x\right)}{\sqrt{2\pi t^3}} \mathbb{1}_{\{0 \le x \le 2\alpha t\}}, \quad t > 0, \tag{4.1}$$

- *A standard Brownian excursion* $\mathbf{e}$ *on* $[0,1]$,
- *A linear Brownian motion* $(\tilde{B}_t^{-(\alpha+\beta)})_{t \ge 0}$ *with drift* $-(\alpha+\beta)$.

Define the process

$$\mathfrak{E}_t = \begin{cases} \sqrt{\tau}\mathbf{e}(\frac{t}{\tau}) + 2\alpha t, & 0 \le t \le \tau, \\ 2\alpha\tau + \tilde{B}_{t-\tau}^{-(\alpha+\beta)}, & \tau \le t \le \tau + \tilde{T}_{\hat{\gamma}}, \end{cases}$$

where $\tilde{T}_{\hat{\gamma}} := \inf\{t \ge 0 : \tilde{B}_t^{-(\alpha+\beta)} = -\hat{\gamma}\}$. *Then,*

$$(X_{t+D} - X_D + \alpha t, \, 0 \le t \le \zeta) \overset{d}{=} (\mathfrak{E}_t, \, 0 \le t \le \tau + \tilde{T}_{\hat{\gamma}}).$$

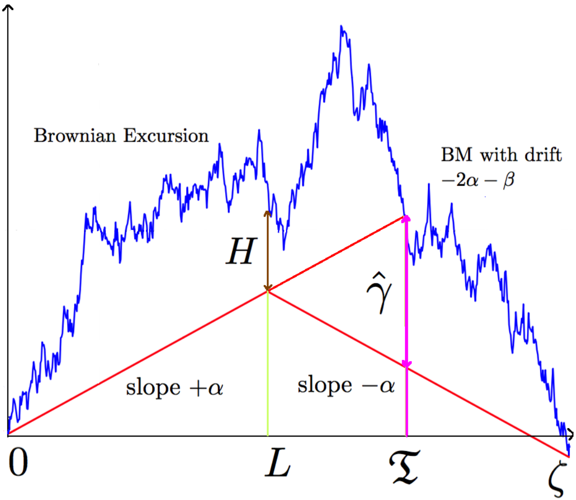

Figure: A generic excursion away from the contact set.

Proposition

The Laplace transform of ζ (the lifetime) is

$$\mathbb{E}[\exp(-\lambda\zeta)] = \frac{4\alpha}{2\alpha + \sqrt{2\lambda + (\alpha+\beta)^2} + \sqrt{2\lambda + (\alpha-\beta)^2}}.$$

In particular, for $\beta = 0$ the density of ζ is

$$z \mapsto 2\alpha \frac{e^{-\frac{\alpha^2 z}{2}}}{\sqrt{2\pi z}} - 2\alpha^2 \overline{\Phi}(\alpha\sqrt{z})$$

where $\overline{\Phi}(x) := \int_x^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$.

Proposition

The Laplace transform of L (the "peak time") is

$$\mathbb{E}[\exp(-\lambda L)] = \frac{4\alpha}{3\alpha + \beta + \sqrt{2\lambda + (\alpha - \beta)^2}}.$$

The density of L is

$$\ell \mapsto 4\alpha \frac{e^{-\frac{(\alpha-\beta)^2 \ell}{2}}}{\sqrt{2\pi\ell}} - 4\alpha(3\alpha + \beta) e^{4\alpha(\alpha+\beta)\ell} \overline{\Phi}(\sqrt{\ell}(3\alpha + \beta)).$$

Proposition

The Laplace transform of $X_{\zeta+D} - X_D$ (the final value) is

$$\mathbb{E}[\exp(-\lambda(X_{\zeta+D} - X_D)] = \frac{4\alpha}{2\alpha + \sqrt{(\alpha+\beta)^2 - 2\lambda} + \sqrt{(\alpha-\beta)^2 + 2\lambda}}.$$

Size-biasing excursions

Theorem

Put

$$W^{\text{straddle}} := (X_{t+G} - X_G,\, 0 \le t \le D - G)$$

for the excursion that contains time zero and

$$W^{\text{generic}} := (X_{t+D} - X_D,\, 0 \le t \le \zeta)$$

for the next excursion; the latter has the same distribution as any other excursion except W^{straddle}*. We have the size-biasing formula*

$$\mathbb{E}\left[F(W^{\text{straddle}})\right] = \mathbb{E}[\zeta]^{-1}\mathbb{E}\left[\zeta F(W^{\text{generic}})\right]$$

for every nonnegative measurable function F*.*

THANK YOU