On Renewal Contact Process

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Based on joint papers with
L.R. Fontes, D. Marchetti, T. S. Mountford, D. Ungaretti

and ongoing work with M. Hilário, D. Ungaretti, D. Valesin

Random Excursions with Jean Bertoin - July 2021

The classical contact process

• $G = (\mathbb{V}, \mathbb{E})$ graph, locally finite. Most classical example $G = \mathbb{Z}^d$.

• A Markov process $\{\xi_t\}_{t\geq 0}$ with values on $\{0,1\}^{\mathbb{V}}$:

$\xi_t(x) = 1$ means x is infected at time t

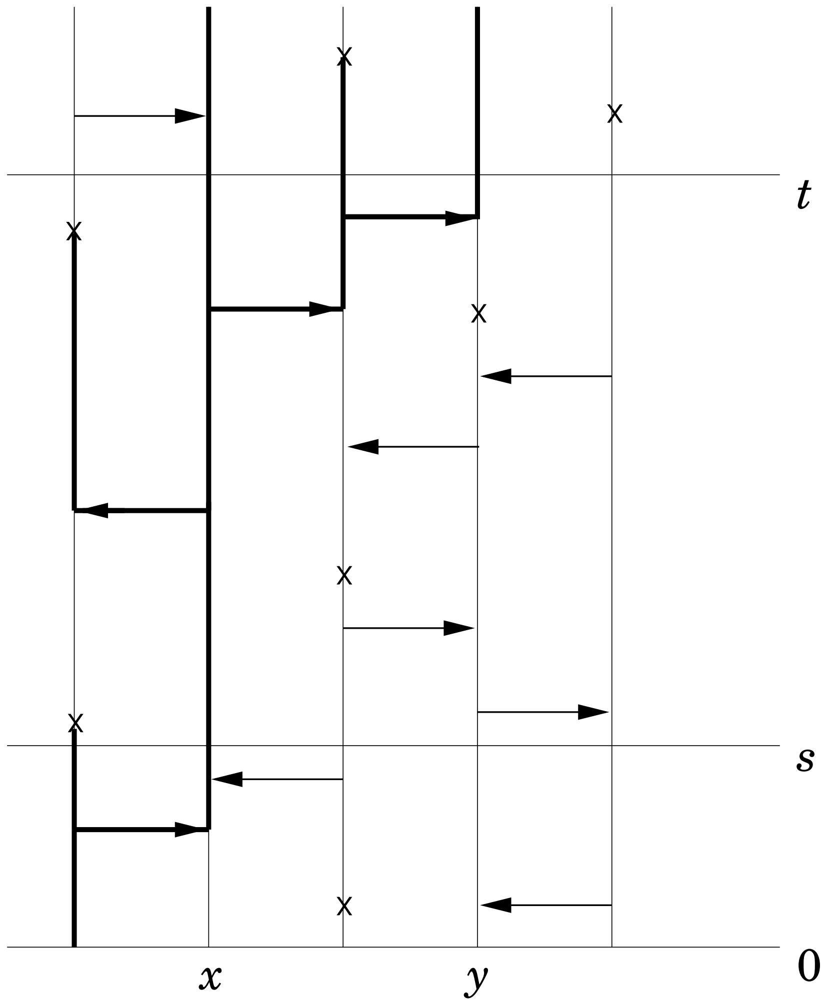
$\xi_t(x) = 0$ means x is healthy at time t

• An infected individual transmits the infection with rate $\lambda > 0$ to each of its healthy neighbors, and heals with rate 1.

Identify ξ_t with $\{x \colon \xi_t(x) = 1\}$ (set of infected individuals at time t)

Model introduced by T. Harris in 1974

The classical contact process



Percolation substructure

Independent P.p.p. $\{\mathcal{N}_{x,y}\}_{x,y}$ $\{\mathcal{N}_x\}_x$

Dynamical phase transition

There exists $\lambda_c \in (0, +\infty)$ so that

- If $\lambda < \lambda_c$ then $\mathbb{P}(\xi_t^{\{0\}} = \underline{0}$ for some $t) = 1$ (subcritical)
- If $\lambda > \lambda_c$ then $\mathbb{P}(\xi_t^{\{0\}} \neq \underline{0}$ for all $t) > 0$ (supercritical)
- $\lambda > \lambda_c \quad \Rightarrow$ positive probability that the infection remains forever

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For more general graphs than $\mathbb{Z}^d$ the supercritical regime splits into at least two:
$0 < \lambda_{1,c} < \lambda_{2,c} < \infty$ (Pemantle (1992), homogeneous tree)

- Weak survival $\quad \lambda \in (\lambda_{1,c},\ \lambda_{2,c})$
- Strong survival $\quad \lambda > \lambda_{2,c}$
- For $G = \mathbb{Z}^d$ these two critical values coincide.

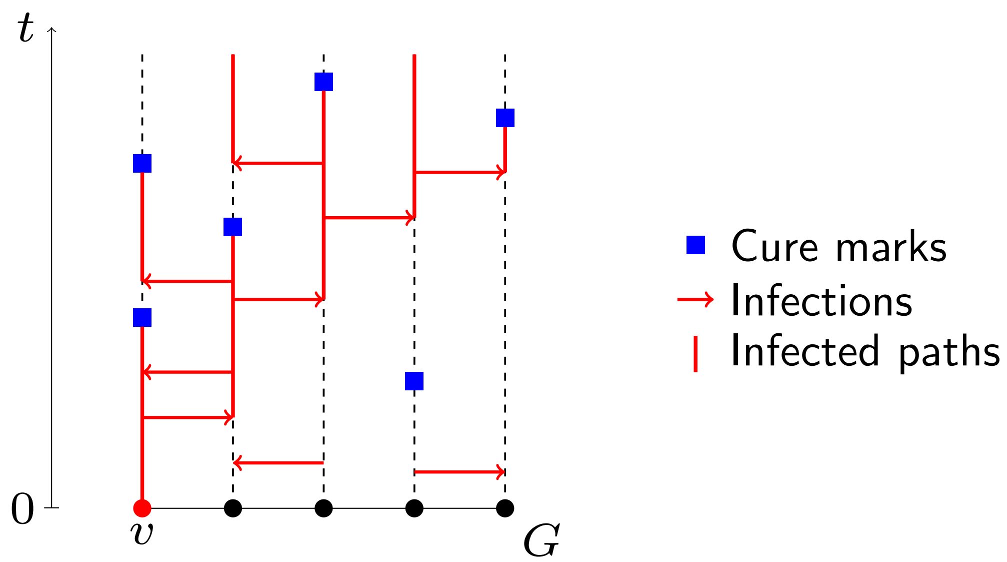
$\lambda > \lambda_c \Rightarrow$ two extremal invariant measures: $\nu_\lambda, \delta_{\underline{0}}$.

There is a huge literature. (Two related monographs by T. Liggett)

The renewal contact process

Same graphical construction. More general point processes

$\{\mathcal{N}_x\}_x$ as recovery and $\{\mathcal{N}_{x,y}\}$ as transmission times.



The process is constructed via *paths* as before.

Basic issue: Survival / Extinction

$$\text{Survival} \iff \text{Percolation}$$

The renewal contact process

Not only Markov property is lost. Other features, like FKG property, might be lost.

A natural setup is that of independent renewal processes (RCP).

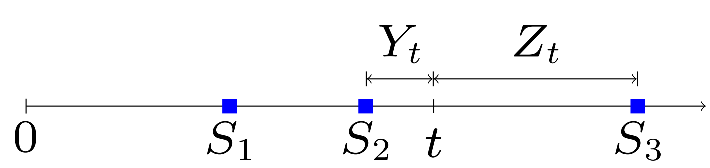
As a first example: $G = \mathbb{Z}^d$

• $\mathcal{N}_{x,y}$ as P.p.p. λ.

• Cure marks given by i.i.d. renewal processes $\{\mathcal{R}_x\}_x$ interarrival distribution μ.
(μ assumed non degenerated.)

(T_i) i.i.d. μ and $\mathcal{R} = \{S_n \colon n \geq 1\}$, with

$$S_n := T_1 + T_2 + \ldots + T_n. \quad (1)$$



Y_t: age process $\qquad$ Z_t: overshooting (or residual) process.

A natural question: For which measures μ do we have $\lambda_c(\mu) = 0$?

In other words,

- For which μ do we have

$$\mathbb{P}(\xi_t^{\{0\}} \neq \underline{0} \text{ for all } t) > 0 \text{ for all } \lambda > 0?$$

In this particular case, monotonicity in λ is of course preserved.

Basic tools: Good control on renewal marks.

Renewal measure: $U(I) := E(\#\text{ renewals on } I)$

Strong Renewal Theorem: Erickson (1970), Chi (2015), Caravenna, Doney (2019)

At least under some good regularity on μ.

Theorem 1. (Fontes, Marchetti, Mountford, V, '19)
If $d \geq 1$ and $\mu(t, +\infty) \geq t^{-\alpha}$ for some $\alpha < 1$ (all t large) plus some regularity conditions, then $\lambda_c(\mu) = 0$.

Theorem 1'. (Fontes, Mountford, Ungaretti, V, '21)
Under the conditions of Theorem 1, complete convergence holds, for any $d \geq 1$. For all $\lambda > 0$, any initial configuration ξ_0 at time 0, ξ_t converges in law, as $t \to \infty$, to

$$\mathbb{P}(\tau = \infty)\delta_{\underline{1}} + \mathbb{P}(\tau < \infty)\delta_{\underline{0}},$$

where $\tau = \inf\{t > 0 \colon \xi_t = 0\}$.

Our assumptions are more general. But include

- $\mu(t, +\infty) = L(t)/t^{\alpha}$ for $\alpha \in (0, 1)$ and $L(\cdot)$ slowly varying at infinity.

Precise assumptions on μ:

A) There exist $1 < M_1 < \infty$, $\epsilon_1 > 0$ and $t_0 \ \in \ (0, \infty)$ such that

$$\forall \ t > t_0, \ \epsilon_1 \int_{[0,t]} s \, \mu(ds) < t\mu(t, M_1 t).$$

B) There exist $1 < M_2 < \infty$, $\epsilon_2 > 0$ and $r_2 < \infty$ so that

$$\forall \ r \geq r_2, \ \ \epsilon_2 \mu[M_2^r, M_2^{r+1}] \leq \mu[M_2^{r+1}, M_2^{r+2}].$$

C) There exist $M_3 < \infty$, $\epsilon_3 > 0$ so that for $t \geq M_3$,

$$t^{-(1-\epsilon_3)} \ \leq \ \mu(t, +\infty) \ \leq \ t^{-\epsilon_3}.$$

Comments: In the special case

$\mu(t, +\infty) = L(t)/t^\alpha$ for $\alpha \in (0, 1)$ and $L(\cdot)$ slowly varying at infinity (*)

one can say much more than Theorem 1 and Theorem 1':

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$$\mu(t, +\infty) = L(t)/t^\alpha \text{ for } \alpha \in (0,1) \text{ and } L(\cdot) \text{ slowly varying at infinity (*)}$$

one can say much more than Theorem 1 and Theorem 1':

- About survival/extinction. Fontes, Gomes, Sanchis '20

$$G = (V, E) \text{ finite and connected and } \mu \text{ as in (*) with } \alpha \in (1/2, 1)$$

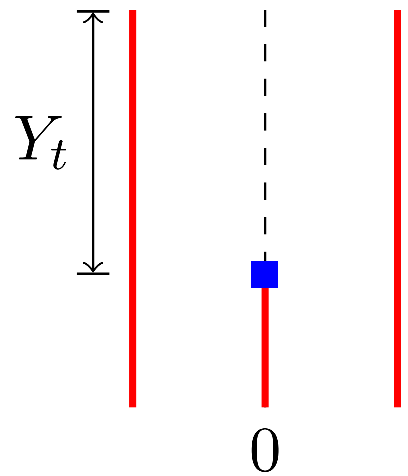
Phase transition on $|V|$: for any fixed $\lambda > 0$,

$$\mathbb{P}_{\lambda,\mu}(\text{infection survives}) \begin{cases} = 0 & \text{if } |V| < V_-(\alpha) := 2 + \frac{2\alpha-1}{(1-\alpha)(2-\alpha)}; \\ > 0 & \text{if } |V| > V_+(\alpha) := \frac{1}{1-\alpha}; \end{cases}$$

Remark: Bounds are tight: $|V_+(\alpha) - V_-(\alpha)| < 1.$

- About the convergence.

$$\mathbb{P}(\xi_t(0) = 1 | \mathcal{R}, \text{ survival}).$$



For large t, $\xi_t(0) = 1$ if there is infection on $[t - Y_t(0), t]$

Theorem 2 (Fontes, Mountford, Ungaretti, V. '21)

(Closeness to determinism)

For RCP on $\mathbb{Z}^d$, let $\mu(t,\infty) = L(t)t^{-\alpha}$. Let $\mathcal{G} = \sigma(\{\mathcal{R}_x\}_x, \tau)$.

- If $\alpha \in (0, 1/2)$ (+ regularity assumptions (SRT)[1]), on $\{\tau = \infty\}$:

$$\lim_{t\to\infty} \left|\mathbb{P}(\xi_t(0) = 1 \mid \mathcal{G}) - \left(1 - e^{-2d\lambda Y_t(0)}\right)\right| = 0, \quad \text{a.s.}$$

- If $\alpha \in (1/2, 1)$, on $\{\tau = \infty\}$:

$$\overline{\lim_{t\to\infty}} \left|\mathbb{P}(\xi_t(0) = 1 \mid \mathcal{G}) - \left(1 - e^{-2d\lambda Y_t(0)}\right)\right| > 0, \quad \text{a.s.}$$

Can give better description here, depending on k s.t. $1 - \alpha \in ((k+2)^{-1}, (k+1)^{-1})$.

[1] Caravenna, Doney

A second question: When do we have $\lambda_c(\mu) > 0$?

Easy fact:

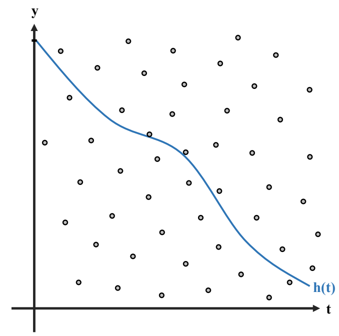
If $\int t^2 \mu(dt) < \infty$ then $\lambda_c(\mu) > 0$ for any $d \geq 1$.

How to improve this?

Hypothesis A: μ has a density f and decreasing hazard rate $h(t) := \frac{f(t)}{\mu(t,+\infty)}$.

What is the role of these assumptions?

• The point process $\mathcal{R}_x$ can be constructed as an increasing function of a P.p.p. on $[0,\infty)^2$. This allows to use FKG inequalities.



As a consequence:

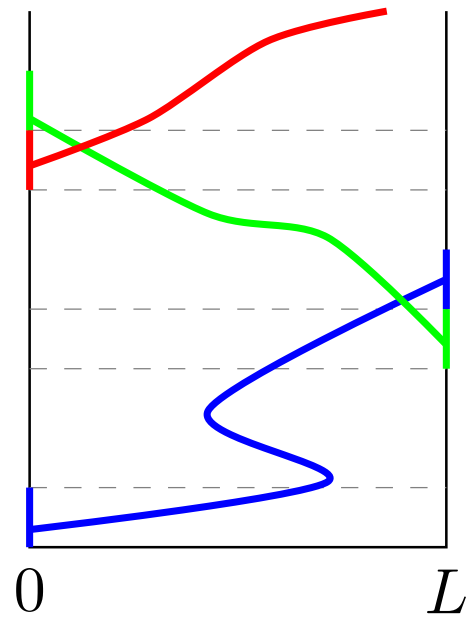
Proposition Assume hypothesis A. Let events $A_1, A_2, \ldots, A_n$ be increasing events on a finite space time rectangle. Then

$$\mathbb{P}(\cap_{i=1}^n A_i) \;\geq\; \prod_{i=1}^n \mathbb{P}(A_i).$$

Theorem 3 (Fontes, Mountford, V, '20)
Let $d = 1$. If μ satisfies Hypothesis A and $\int t^{\alpha}\mu(dt) < \infty$ for some $\alpha > 1$, then $\lambda_c(\mu) > 0$.

Our arguments relied on putting together distinct crossing paths. They require $d = 1$.

- Allows to use arguments that show similarity with known RSW estimates in percolation.



Theorem 4 (Fontes, Mountford, Ungaretti, V.'21)

Consider the RCP on $\mathbb{Z}^d$, and assume that μ satisfies

$$\int_1^\infty x \exp\left[\theta(\ln x)^{1/2}\right]\mu(\mathrm{d}x) < \infty \quad \text{for some } \theta > \sqrt{(8\ln 2)d}.$$

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Then $\lambda_c(\mu) > 0$.

Open: Exact relation between $\lambda_c(\mu) > 0$ and finite first moment ?

Contact process with dynamic edges - Linker, Remenik '20

A dynamic *environment* process $\zeta_t \in \{0,1\}^{E(\mathbb{Z}^d)}$.

Given ζ_t, one has the *infection* process $\eta_t \in \{0,1\}^{\mathbb{Z}^d}$:

- Contact process with transmissions only through the open edges of ζ.

Fix $v > 0$ and $p \in (0,1)$. Independently of everything else, each edge e is assigned an initial state $\zeta_0(e)$ Bernoulli with parameter p, and independently updates its state as follows:

$$0 \longrightarrow 1 \quad \text{at rate} \quad vp,$$
$$1 \longrightarrow 0 \quad \text{at rate} \quad v(1-p).$$

$\{\zeta_t\}_{t\geq 0}$ dynamic bond percolation on $\mathbb{Z}^d$ with parameter p and rate v.

The contact process η_t depends on the rate $\lambda > 0$ and on the underlying ζ_t.

At each site x, the state $\eta_t(x)$ evolves as follows:

$$1 \longrightarrow 0 \quad \text{at rate} \quad 1,$$
$$0 \longrightarrow 1 \quad \text{at rate} \quad \lambda \sum_{y\sim x} \zeta_t(xy)\eta_t(y). \tag{2}$$

Linker and Remenik proved several results for this process. One of those only for $\mathbb{Z}$.

This is the one we now extend to $d \geq 2$ in (a) below.

$$\lambda_c(v,p) := \inf\{\lambda > 0;\ \mathbb{P}_{v,p,\lambda}(\eta_t^{\{0\}} \not\equiv 0,\ \forall t > 0) > 0\}.$$

Theorem 5 (Hilário, Ungaretti, Valesin, V) - in preparation

Consider the CPDE on $\mathbb{Z}^d$ with $d \geq 1$.

(a) For all $p < p_c(d)$, $\lim_{v\to 0} \lambda_c(v,p) = \infty$

(b) For all $p > p_c(d)$ and all v, $\lambda_c(v,p) < \infty$.

Proof ideas (for Th.1, Th.2 and Th.4)

We use some uniform bounds:

- Under Assumptions of Theorem 1

$$\text{(i)} \quad \inf_{t \geq 1} \mathbb{P}(\mathcal{R} \cap [t, Kt] = \varnothing) > 0, \text{ for each } K < \infty.$$

(ii) There is $\epsilon > 0$ such that for large t

$$\mathbb{P}(\mathcal{R} \cap [t, t + t^{\epsilon}] = \varnothing) \geq 1 - t^{-\epsilon}.$$

$\Rightarrow$ Conditioned on survival, w.h.p. for all large t, there exists x_t with $|x_t| \leq (\ln t)^3$ s.t. $\xi_s(x_t) = 1$ for all $s \in [t/2.t]$.

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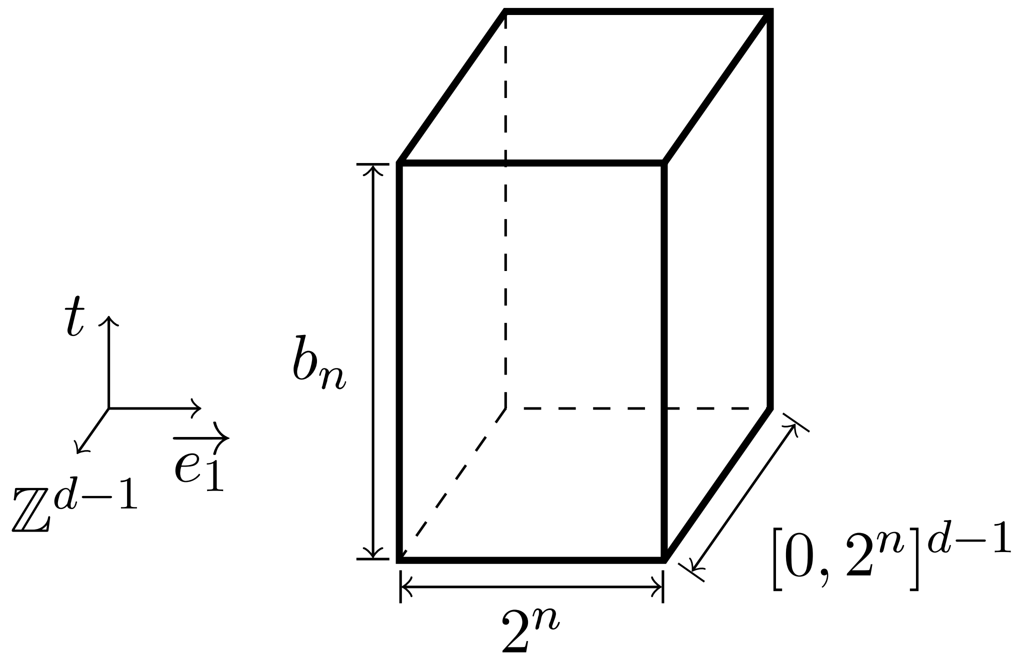
- If $\int x f(x) \mu(\mathrm{d}x) < \infty$, with $f : [0, \infty) \to [0, \infty)$ non-decreasing, then

$$\sup_{t \geq 0} \mathbb{P}(\mathcal{R} \cap [t, t + u] = \varnothing) \leq \frac{C}{f(u)}.$$

Proof Ideas

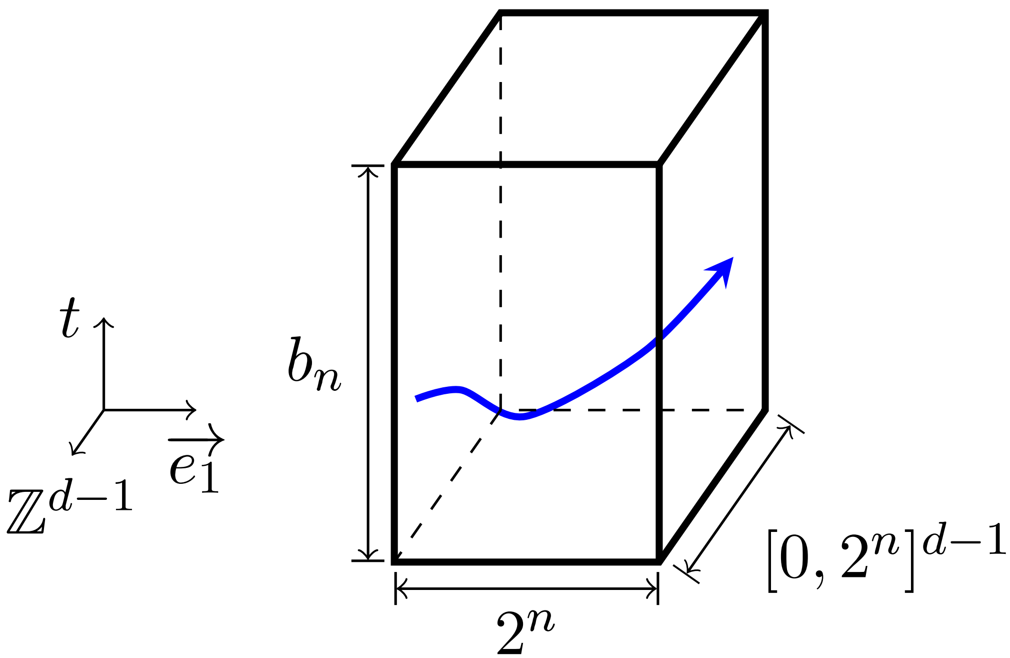
Boxes $B_n := [0, 2^n]^d \times [0, b_n]$, with b_n chosen later (fastly growing)



Four basic events:

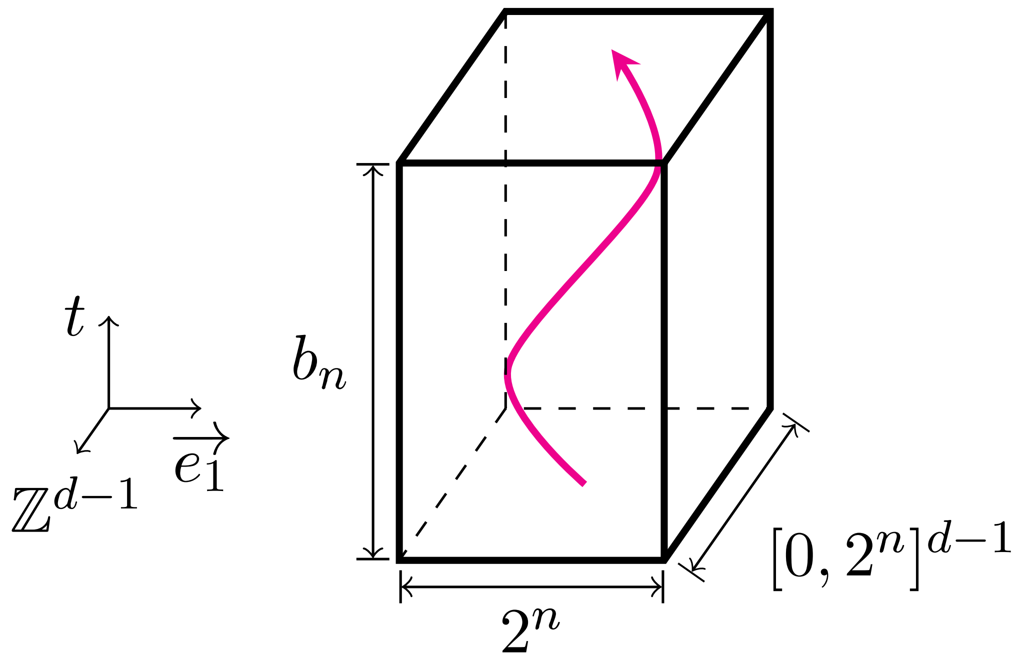
Proof Ideas

Spatial crossing $S_j(B)$



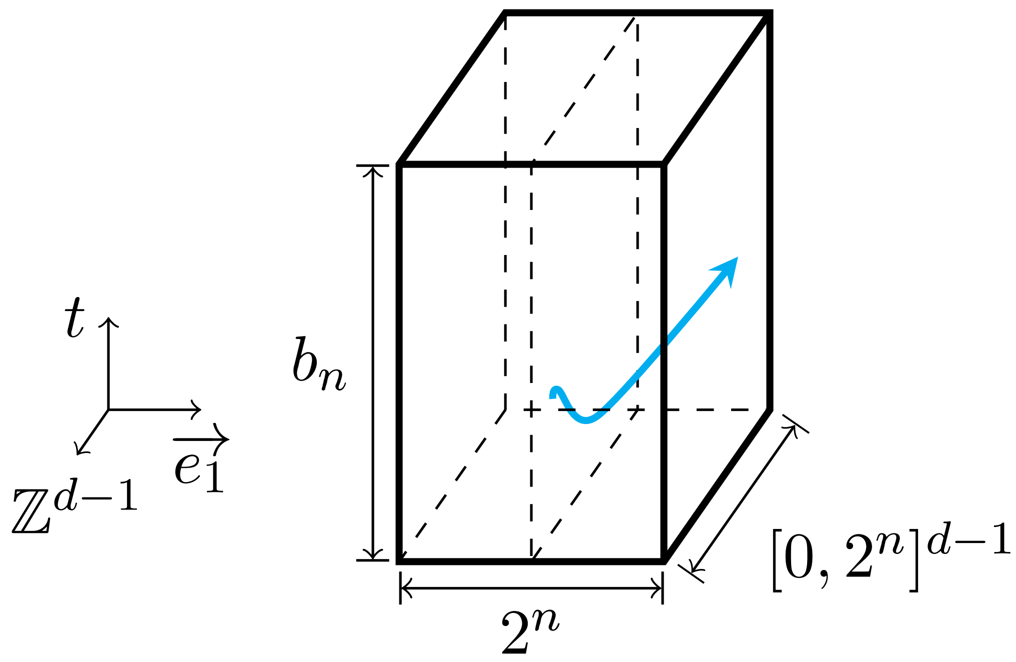
Proof Ideas

Temporal crossing $T(B)$



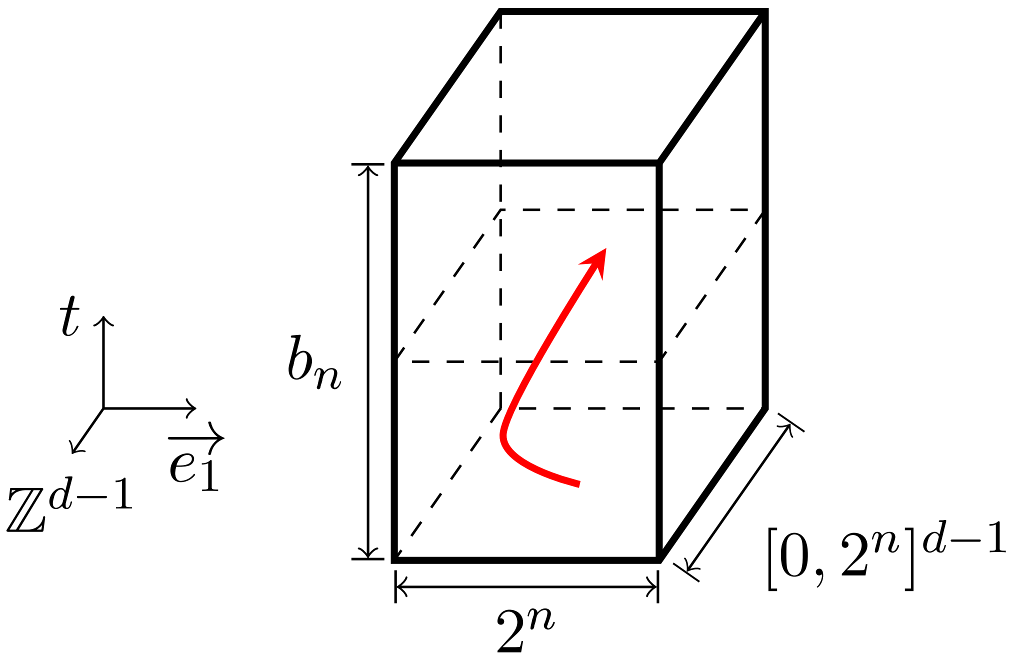
Proof Ideas

Spatial half-crossing. $S_j(\tilde{B}_j)$



Proof Ideas

Temporal half-crossing. $\tilde{T}(B)$



Want to show that for $\lambda > 0$ small enough,

$$\mathbb{P}(\tau^0 = \infty) \leq \mathbb{P}(T(B_n)) + 2d \cdot \mathbb{P}(S_1(\tilde{B}_1(n))) \longrightarrow 0.$$

Proof Ideas

- Define uniform quantities. (x,t): translations on $\mathbb{Z}^d \times \mathbb{R}_+$.

$$s_n := \sup \hat{\mathbb{P}}(S_j((x,t)+B_n)), \qquad t_n := \sup \hat{\mathbb{P}}(T((x,t)+B_n)),$$
$$h_n := \sup \hat{\mathbb{P}}(S_j((x,t)+\tilde{B}_j(n))), \qquad \tilde{t}_n := \sup \hat{\mathbb{P}}(\tilde{T}((x,t)+B_n)),$$

The $\sup$: over all $(x,t) \in \mathbb{Z}^d \times \mathbb{R}_+$ and all product renewals $\hat{\mathbb{P}}$ interarrival μ starting at (possibly different) time points less than zero.

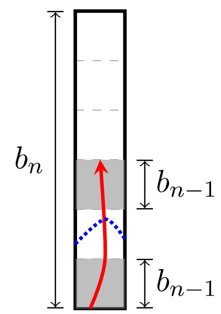
- Immediate relations:

$$t_n \le \tilde{t}_n \qquad \text{(by inclusion)}$$
$$s_n \le h_n \qquad \text{(by inclusion)}$$
$$s_n \le h_n^2 \qquad \text{(by spatial independence)}$$

Proof Ideas

Lemma (Temporal half-crossing) If $b_n \geq 6b_{n-1}$ and $\int x f(x)\,\mu(\mathrm{d}x) < \infty$:

$$\tilde{t}_n \leq \frac{C2^{dn}}{f(b_{n-1})} + (3^d t_{n-1} + 2d \cdot 3^{d-1} h_{n-1})^2.$$



- If not enough cures: Error term
- If enough cures: Independent $[0, 2^n]^d \times b_{n-1}$ boxes!

Lemma (Spatial half-crossing) - Easier by the independence across space

$$h_n \leq 4 \cdot 36^{d-1} \cdot \left\lceil \frac{b_n}{b_{n-1}} \right\rceil^2 \cdot (h_{n-1} + \tilde{t}_{n-1})^2.$$

Proof Ideas Defining $u_n := h_n + \tilde{t}_n$, we have

$$u_n \leq C(d) \cdot (b_n/b_{n-1})^2 \cdot u_{n-1}^2 + \frac{C2^{dn}}{f(b_{n-1})}.$$

Prove by induction that good choice of f and suitable b_n there exists n_0 s.t if $n \geq n_0$

$$u_n \leq 2^{-dn} \quad \text{implies} \quad u_{n+1} \leq 2^{-d(n+1)}.$$

Good choices for f and b_n:

$$f = e^{\theta(\ln x)^{1/2}} \mathbb{1}\{x \geq 1\} \quad \text{and} \quad b_n = e^{(\alpha/\theta)^2 n^2},\ 2d\ln 2 < \alpha < \sqrt{\frac{\theta^2 d \ln 2}{2}}$$

Prove:

$$u_{n_0} \leq 2^{-dn_0} \quad \text{if } \lambda \text{ small enough.}$$

Question: How to compare $\mathcal{R}(\mu)$ and $\mathcal{R}(\nu)$? Coupling?

There is a classical general result by H. Rost. (How to use it???)

Thanks!

Happy Birthday, Jean!